ANALYSIS OF LMS ALGORITHM WITH DELAYED COEFFICIENT ADAPTATION FOR SINUSOIDAL REFERENCE

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ABSTRACT

In this paper, we analyse the stability of LMS algorithms with delayed coefficient adaptation, such as FxLMS and DLMS algorithms, when sinusoidal references are used. We demonstrate that in two common situations, that are two in-phase and quadrature sinusoidal components and a tapped delay line with a large number of weights, the adaptive system turns to be linear and time-invariant. Without the need of any independence assumption, stability is then analysed in those two situations and an upper bound for the adaptation step size is derived.

1 INTRODUCTION

In some practical applications, for several reasons, the adaptation error used to update the coefficients of the Least Mean Squares (LMS) algorithm may not be available when needed, but several time instants later. Some examples of this situation are Viterbi decoding, adaptive reference echo cancellation and implementation of adaptive algorithms using parallel architectures. In these cases the filter coefficients have to be updated with a delayed version of the error signal. This modified version of LMS algorithm is known as "Delayed LMS" algorithm (DLMS).

A similar situation is also encountered in active control of noise or vibration. In this case, the output of the adaptive filter goes through a secondary-path transfer function before being subtracted from the primary signal to generate the error signal. The alternative form of the LMS algorithm for active control is the so-called filtered reference LMS or filtered-x LMS (FxLMS) [1]. In the special case where this secondary-path transfer function is just a pure delay, the FxLMS algorithm becomes the DLMS algorithm.

The convergence, the stability properties and the steady-state behaviour of DLMS algorithm have been investigated for reference signals with Gaussian distribution [2, 3]. Also, the case of spherically invariant input processes has been analysed for the DLMS algorithm [4]. However, most of the applications in active control aim to cancel tonal disturbances, since they are the most annoying as well as the easiest to obtain a good reference and, so, to cancel. Moreover, in both previous analysis it was necessary to assume independence between reference data and filter coefficients in order to simplify the analysis. Nevertheless, independence assumption is considered valid only when convergence is slow.

In this paper, we analyse the DLMS algorithm when sinusoidal references are used, without making any independence assumption. In particular, we study two common cases where the global system turns to be linear and time-invariant, and we obtain a bound for the step size that guarantees the stability of the algorithm.

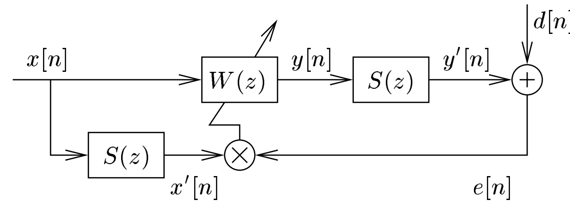

Figure 1: Block diagram of FxLMS system.

2 FXLMS WITH SINUSOIDAL REFERENCE

In figure 1 is depicted a block diagram of the FxLMS algorithm. Due to the presence of the secondary-path transfer function, $S(z)$, a new signal, $x'[n]$, has to be used in the adaptation of the filter coefficients. This signal is the result of filtering the reference, $x[n]$, by a model of the secondary-path transfer function, $\widehat{S}(z)$. In our analysis, we consider perfect modelling of this transfer function, that is, $\widehat{S}(z) = S(z)$.

Thus, the FxLMS algorithm is given by the following equations:

$$\begin{aligned} y[n] &= \mathbf{w}^T[n]\,\mathbf{x}[n] \\ y'[n] &= s[n] * y[n] \\ e[n] &= d[n] - y'[n] \\ \mathbf{x}'[n] &= s[n] * \mathbf{x}[n] \\ \mathbf{w}[n+1] &= \mathbf{w}[n] + 2\mu e[n]\,\mathbf{x}'[n] \end{aligned} \tag{1}$$

where $s[n]$ is the impulse response of the secondary transfer function, and $*$ denotes convolution.

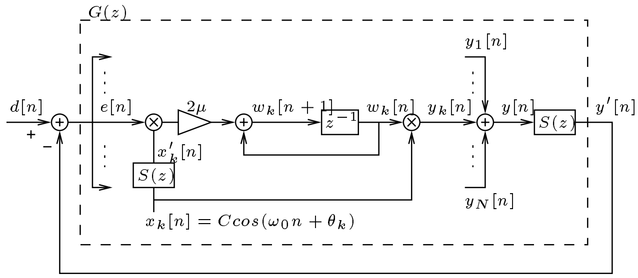

Figure 2: Flow diagram of FxLMS algorithm.

According to Elliott and Nelson [1], an empirical bound for the step size that can be used in this algorithm with white noise reference signals is given by

$$2\mu_{max} \approx \frac{2}{P_{x'}\left(L+\Delta\right)} \tag{2}$$

where L is the order of the adaptive filter, $P_{x'}$ is the power of the filtered reference and Δ is the overall delay of the secondary path. Similar results were obtained analytically for the DLMS algorithm with white noise references in [2, 3].

When the reference signal is sinusoidal, each of the elements of the reference signal vector, $\mathbf{x}[n]$, can be expressed as

$$\begin{aligned} x_k[n] &= C\cos\left(\omega_0 n+\theta_k\right) \\ &= \frac{C}{2}\left(e^{j\omega_0 n}e^{j\theta_k}+e^{-j\omega_0 n}e^{-j\theta_k}\right) \end{aligned}$$

The FxLMS signal flow diagram, according to the set of equations (1), is showed in detail for one of these elements $x_k[n]$ in figure 2. From this diagram, the z transform of the output $y'[n]$ can be computed. This analysis is similar to the one done by Glover for the LMS algorithm in [5].

Thus, the kth weight is

$$\begin{aligned} W_k(z) &= C\mu U(z)\left[E\left(ze^{-j\omega_0}\right)S\left(e^{j\omega_0}\right)e^{j\theta_k}\right. \\ &\quad \left.+E\left(ze^{j\omega_0}\right)S\left(e^{-j\omega_0}\right)e^{-j\theta_k}\right] \end{aligned}$$

where $U(z)=\frac{1}{z-1}$ is the z transform of the step function. The contribution of the kth weight to the output of the adaptive filter is given by

$$Y_k(z)=\frac{C}{2}\left[W_k\left(ze^{-j\omega_0}\right)e^{j\theta_k}+W_k\left(ze^{j\omega_0}\right)e^{-j\theta_k}\right]$$

Substituting for W_k and rearranging yields

$$\begin{aligned} &Y'(z)=Y(z)S(z)=\left(\sum_{k=1}^{L}Y_k(z)\right)S(z)= \\ &=\frac{C^2\mu L}{2}\left\{U\left(ze^{-j\omega_0}\right)S\left(e^{-j\omega_0}\right)+\right. \\ &\left.+U\left(ze^{j\omega_0}\right)S\left(e^{j\omega_0}\right)\right\}E(z)S(z)+ \end{aligned} \tag{3}$$

$$\begin{aligned} &+\frac{C^2\mu}{2}\left\{E\left(ze^{-j2\omega_0}\right)U\left(ze^{-j\omega_0}\right)S\left(e^{j\omega_0}\right)\sum_{k=1}^{L}e^{j2\theta_k}+\right. \\ &\left.+E\left(ze^{j2\omega_0}\right)U\left(ze^{j\omega_0}\right)S\left(e^{-j\omega_0}\right)\sum_{k=1}^{L}e^{-j2\theta_k}\right\}S(z) \end{aligned} \tag{4}$$

The terms in (3) represent the time-invariant part of the response from $E(z)$ to $Y'(z)$, since only frequencies of $E(z)$ appear at the output. On the other hand, the terms in (4) are time-varying, since they introduce unwanted frequency shifted components of $E(z)$ at the output $Y'(z)$.

In the next sections, two common situations where the time-varying terms are zero or neglectable are analysed. When this happens, the response from $E(z)$ to $Y'(z)$ is linear and time-invariant, and we can define the open loop transfer function $G(z)$ from figure 2:

$$\begin{aligned} G(z) &= \tfrac{Y'(z)}{E(z)}=\tfrac{L}{2}C^2\mu\left\{\tfrac{S\left(e^{-j\omega_0}\right)}{ze^{-j\omega_0}-1}+\tfrac{S\left(e^{j\omega_0}\right)}{ze^{j\omega_0}-1}\right\}S(z) \\ &= LC^2\mu\left|S\left(e^{j\omega_0}\right)\right|\left\{\tfrac{\cos\left(\omega_0-\phi_{\omega_0}\right)z-\cos\left(\phi_{\omega_0}\right)}{z^2-2\cos(\omega_0)z+1}\right\}S(z) \end{aligned} \tag{5}$$

where $\phi_{\omega_0}=\angle S\left(e^{j\omega_0}\right)$. The closed loop transfer function, from the primary input $D(z)$ to the error output $E(z)$, can be easily obtained from $G(z)$ as

$$H(z)=\frac{E(z)}{D(z)}=\frac{1}{1+G(z)} \tag{6}$$

In the specific case where the adaptive system is linear and time-invariant, its maximum step size can be found analysing the stability of its transfer function, $H(z)$, without the need of any independence assumption.

3 I/Q SINUSOIDAL REFERENCE

Consider the case where the reference is composed of two sinusoidal, in-phase and quadrature (I/Q) components:

$$\mathbf{x}[n]=\begin{bmatrix} x_1[n] \\ x_2[n]\end{bmatrix}=\begin{bmatrix} C\cos\left(\omega_0 n+\theta\right) \\ C\sin\left(\omega_0 n+\theta\right)\end{bmatrix}$$

In this case, $L=2$, $\theta_1=\theta$, $\theta_2=\theta-\frac{\pi}{2}$, and $\sum_{k=1}^{L}e^{\pm j2\theta_k}=0$. Hence, the time-varying terms in (4) are exactly zero.

Let the secondary-path transfer function be just a delay, with a possible change in amplitude, $S(z)=Az^{-\Delta}$. Thus, the open loop transfer function from equation (5) turns into

$$\begin{aligned} G(z) &= 2C^2A^2\mu\frac{\cos\left(\omega_0\left(\Delta+1\right)\right)z-\cos\left(\omega_0\Delta\right)}{z^{\Delta}\left(z^2-2\cos\left(\omega_0\right)z+1\right)} \\ &= K\frac{Num(z)}{Den(z)} \end{aligned} \tag{7}$$

with $K=2C^2A^2\mu$, $Num(z)=\cos\left(\omega_0\left(\Delta+1\right)\right)z-\cos\left(\omega_0\Delta\right)$ and $Den(z)=z^{\Delta}\left(z^2-2\cos\left(\omega_0\right)z+1\right)$.

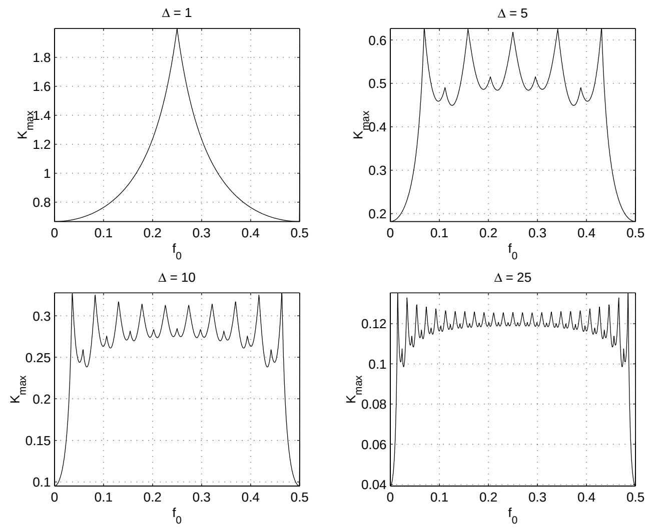

Figure 3: Bound for $K = 2C^2A^2\mu$ as a function of f_0, for Δ=1, 5, 10 and 25.

The closed loop transfer function can then be expressed

$$H(z) = \frac{Den(z)}{Den(z) + K\,Num(z)} \tag{8}$$

Since $Num(z)$ and $Den(z)$ are polynomials in z, it is clear from equation (8) that the poles of $G(z)$ are the zeros of $H(z)$. Hence, the adaptive system with sinusoidal reference behaves as a notch filter at the reference frequency, as $e^{\pm j\omega_0}$ are poles of $G(z)$ and zeros of $H(z)$. On the other hand, the poles of $H(z)$ are the roots of the characteristic equation

$$Den(z) + K\,Num(z) = 0 \tag{9}$$

Whenever all of this roots have amplitude less than one, the adaptive system will be stable, that is to say, it will converge. Analysing the root locus of the characteristic equation (9) it is possible to obtain an upper bound for parameter K which ensures stability. That is, the system will be stable only for values of K in the range from 0 to some bound $K_{max}(\omega_0, \Delta)$, dependent with both, frequency and delay. In figure 3 the bound for K is plotted as a function of frequency, for some specific values of the delay.

Observe that the LMS algorithm is the particular case of no delay, that is, $\Delta = 0$, and $S(z) = 1$. It is easy to check that $K_{max}(\omega_0)|_{\Delta=0} = 2$, that is, $\mu_{LMS} < \frac{1}{C^2}$, in accordance with results in [5].

Since the exact frequency of the reference may be unknown prior to the operation of the adaptive system, or simply, it could be changing, it is useful to obtain a bound for the adaptation step size which ensures stability of the system whatever the frequency of the reference is. It can be seen from figure 3 that, for a given delay Δ, the minimum value of K_{max} (which will ensure stability for any frequency ω_0) is obtained when $\omega_0 \to 0$ or $\omega_0 \to \pi$. When $\omega_0 \to 0$, system stability is lost because one of the poles of $H(z)$ moves outside the unit circle crossing $z = 1$. When $\omega_0 \to \pi$, the crossing point is $z = -1$. The bound for K can be found from equation (9) with the conditions $\omega_0 \to 0$, $z = 1$ or $\omega_0 \to \pi$, $z = -1$. Thus,

$$\begin{aligned} K_{max}(\Delta) &= \lim_{\omega_0 \to 0} -\left.\frac{Den(z)}{Num(z)}\right|_{z=1} \\ &= \lim_{\omega_0 \to \pi} -\left.\frac{Den(z)}{Num(z)}\right|_{z=-1} \\ &= \frac{2}{2\Delta + 1} \end{aligned} \tag{10}$$

Therefore, the upper bound for the adaptation step size for any frequency as a function of the delay is given by

$$2\mu_{max} = \frac{1}{P_{x'}(2\Delta + 1)} \tag{11}$$

where $P_{x'} = \frac{C^2A^2}{2}$ is the power of the reference signals.

However, as can be seen from figure 3, this bound is very restrictive for most of the frequencies: those that are not very close to 0 or π. Working with frequencies in the range $\frac{2}{\Delta} < \omega_0 < \pi - \frac{2}{\Delta}$, a less restrictive bound can be given

$$K_{max}(\Delta) \approx \frac{5}{2\Delta + 1} \tag{12}$$

$$2\mu_{max} \approx \frac{5}{2P_{x'}(2\Delta + 1)} \tag{13}$$

4 TDL SINUSOIDAL REFERENCE

It is also interesting to analyse the case of a tapped delay line (TDL) sinusoidal reference. In this case $\theta_k = \theta - \omega_0(k-1)$, and

$$\sum_{k=1}^{L} e^{\pm j2\theta_k} = e^{\pm j(2\theta - \omega_0(L-1))} \frac{\sin(\omega_0 L)}{\sin(\omega_0)} \tag{14}$$

When $\omega_0 = \frac{i\pi}{L}$, with i being an integer, expression (14) is zero and the adaptive system is again linear and time-invariant. Also, when the order L of the adaptive filter is large enough, (14) approaches 0, and the time-varying terms in (4) are neglectable [5]. In these cases, the open loop transfer function for the same secondary path of the previous section, $S(z) = Az^{-\Delta}$, becomes

$$G(z) = LC^2A^2\mu \frac{\cos(\omega_0(\Delta+1))\,z - \cos(\omega_0\Delta)}{z^{\Delta}(z^2 - 2\cos(\omega_0)\,z + 1)} \tag{15}$$

Observe that the open loop transfer function obtained is almost the same as in previous section (see eq. (7)), but for a scale factor of $\frac{L}{2}$. Expressing $G(z)$ again as $K\frac{Num(z)}{Den(z)}$, we only need to redefine $K = LC^2A^2\mu$, since polynomials $Num(z)$ and $Den(z)$ are exactly the same. Therefore, previous stability analysis, as a function of parameter K, of the adaptive system holds, and results (10) and (12) are still valid with the new definition for K. So, the maximum step size for any frequency is now

$$2\mu_{max} = \frac{2}{LP_{x'}\left(2\Delta + 1\right)} \tag{16}$$

whereas the less restrictive bound when $\frac{2}{\Delta} < \omega_0 < \pi - \frac{2}{\Delta}$ is given by

$$2\mu_{max} \approx \frac{5}{LP_{x'}\left(2\Delta + 1\right)} \tag{17}$$

Next, we consider stability for the reference frequency $\omega_0 = \frac{\pi}{2}$, as a representative case. This is the middle point in graphs of figure 3. It can be demonstrated applying root locus theory to the characteristic equation (9) that the maximum K for a stable adaptive system is given by

$$K_{max}|_{\omega_0 = \frac{\pi}{2}} = 2\cos\left(\frac{\pi\left\lfloor\frac{\Delta}{2}\right\rfloor}{2\left\lfloor\frac{\Delta}{2}\right\rfloor + 1}\right) \tag{18}$$

where $\lfloor a \rfloor$ is the "floor" or "greatest integer" function, defined as the greatest integer less than or equal to a. When Δ is large, equation (18) can be approximated as

$$K_{max}|_{\omega_0 = \frac{\pi}{2}} \approx \frac{\pi}{\Delta + 1}$$

and, consequently,

$$2\mu_{max}|_{\omega_0 = \frac{\pi}{2}} = \frac{2}{LP_{x'}}\cos\left(\frac{\pi\left\lfloor\frac{\Delta}{2}\right\rfloor}{2\left\lfloor\frac{\Delta}{2}\right\rfloor + 1}\right) \approx \frac{2\pi}{LP_{x'}\left(\Delta + 1\right)} \tag{19}$$

This result is in close agreement with the ones obtained in previous works by Elliott [6] and Morgan [7] for sinusoidal references with frequency $\omega_0 = \frac{\pi}{2}$.

From equations (16), (17) and (19), it is clear that the maximum step size for sinusoidal reference signals is inversely proportional to the product between the order of the filter and the delay, $L\Delta$. Comparing this result with expression (2), it can be seen that the bound for sinusoidal reference is much more restrictive than for white noise reference.

As an example, consider the case of a reference frequency $\omega_0 = \frac{\pi}{10}$, an adaptive filter order of $L = 10$ samples and a delay in the secondary path of $\Delta = 25$ samples. The primary signal consists of a sinusoid at the reference frequency with random initial phase plus a white noise with a power level $20\,dB$ below the sinusoidal component. The mean squared error (MSE, $\xi\left[n\right]$) for two different values of the adaptation step size was obtained averaging 1000 realizations of the adaptive system. The learning curves obtained are plotted in figure 4. The solid line curve corresponds to the adaptation step size given in eq. (17), whereas the dotted one corresponds to an adaptation step size 18 % greater. Clearly, in the first case convergence is obtained while in the second one the adaptive algorithm diverges. The oscillations in the MSE are due to the complex poles of the equivalent transfer function of the adaptive system.

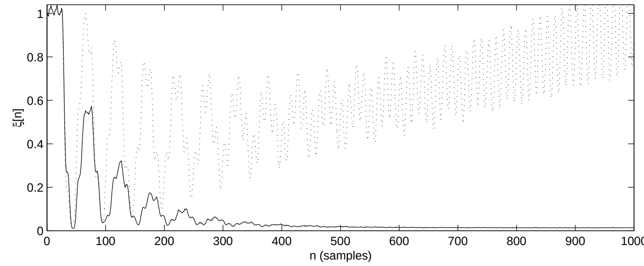

Figure 4: Example learning curves.

5 CONCLUSIONS

In this paper we have analysed convergence of FxLMS and DLMS adaptive systems with sinusoidal reference signals. The equivalent transfer function has been obtained in two situations where the adaptive system proves to be linear and time-invariant: in-phase and quadrature sinusoidal components and tapped delay line with large number of filter weights. Stability of the transfer function has been investigated, and an upper bound for the adaptation step size has been derived in eqs. (16) and (17). This upper bound is more restrictive than the one obtained in previous works for white noise references.

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