TRUNCATED ORTHOGONAL EXPANSIONS OF RECURRENT SIGNALS: EQUIVALENCE TO A PERIODIC TIME-VARIANT FILTER

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ABSTRACT

In this work we show that orthogonal expansions of recurrent signals like electrocardiograms with a reduced number of coefficients can be considered as a periodic time-variant filter. Instantaneous impulse and frequency responses are analyzed for two cases: estimation of the coefficients with inner product and adaptive estimation with LMS algorithm.

1. INTRODUCTION

Orthogonal expansion is a very well-known technique for signal analysis. It is based on the decomposition of the signal in a linear combination of simple and elementary functions [1]. An appropriate choice of the orthogonal functions achieves a signal representation where each coefficient contributes with independent and complementary information. For example, frequency components of the Fourier transform, instantaneous signal values for identity transform, localized frequency components using wavelet transform, etc.

Orthogonal functions that achieve a good energy concentration are especially useful when analyzing a reduced number of coefficients in several applications: data compression [5], parameter extraction for pattern recognition, monitoring [4], etc..

In this work we show that the effect of using a reduced number of coefficients in orthogonal expansions of recurrent signals, like electrocardiograms (ECG), can be described as a periodic time-variant filter. We analyze two different ways for estimating the coefficients: inner product and adaptive estimation with the LMS algorithm. Both methods are analyzed with time and frequency domain responses.

2. INNER PRODUCT

Inner product (IP) is the most common way for estimating the orthogonal expansion coefficients of signals with high values of signal to noise ratio. IP is the solution to the problem of minimizing the mean square error between the original signal and a reduced linear combination of basis functions. When all N basis functions of the signal space are used in the expansion, the signal energy is completely represented, and the system can be considered as the identity function. When the number of functions is reduced to a fraction $p < N$, some signal components are discarded. This behavior can be seen as applying a filter to the signal.

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The input $x[n]$ output $y[n]$ equation of the IP estimation system with a reduced number of coefficients $p < N$ for the *k-th* occurrence of a recurrent signal $x[n]$ can be written as

$$y[(k-1)N+n] = \sum_{i=0}^{p-1} c_i^{k-1}\,\widetilde{\Phi}_i[n], \qquad 0<n<N-1 \quad (1)$$

where $\widetilde{\Phi}_i[n]$ is the periodic extension of each basis function $\Phi_i[n]$ and c_i^{k-1} are the coefficients of the inner product of the signal from the previous occurrence (applying causality to the filter)

$$c_i = \sum_{m=0}^{N-1} \widetilde{\Phi}_i[m]\,x[(k-2)N+m]; \qquad 0<i<p-1. \quad (2)$$

The input-output equation is obtained by substituting (2) in (1)

$$y[(k-1)N+n]=\sum_{m=0}^{N-1} x[(k-2)N+m]\,r[m,n]; \qquad 0<n<N-1 \quad (3)$$

where $r[m,n] = \sum_{i=0}^{p-1} \widetilde{\Phi}_i[m]\widetilde{\Phi}_i[n]= \widetilde{\Phi}^T[m]\widetilde{\Phi}[n]$ and $\widetilde{\Phi}[n]$ is the vector of basis functions at time n. The output at instant n can be seen as a linear combination of input samples from the last occurrence (N samples delay) with time-varying values of the basis functions; that is, IP can be described as a time-variant filter, but periodic because $r[m,n] = r[m+N,n] = r[m,n+N]$. In order to find the instantaneous impulse responses, (3) can be written as a linear convolution

$$y[(k-1)N+n] = \sum_{m=-\infty}^{\infty} x[m]h[n-m,n] \quad (4)$$

of the input signal $x[n]$ with the N finite duration impulse responses $\{h[m,n];\ n=0,1,\ldots,N-1\}$

$$h[m,n]=\begin{cases} r[n-m+N,n] & m=n+1, n+2, \ldots, n+N \\ 0 & \text{elsewhere} \end{cases}. \quad (5)$$

In order to quantify this filtering effect, impulse and frequency responses are studied depending on the number of functions p. The optimal Karhunen-Loève (KL) transform [7] of ECG signals is used as an example, but other orthogonal expansions can be considered without loss of generality. The basis functions are estimated from a training set of ECG signals of MIT-BIH Arrythmia and ESC-STT databases (resampled to 360 Hz). In Figure 1 impulse responses $h[m,n]$ are shown for several values of n within each occurrence when p=30 functions are used in the expansion.

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The impulse responses are finite with duration N=430 samples (heartbeat length).

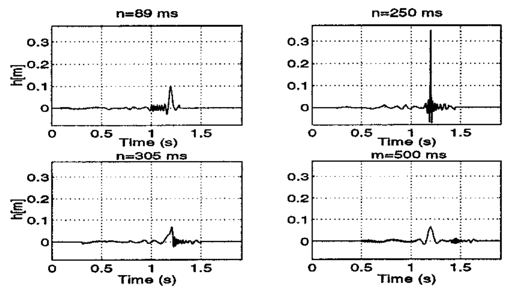

Figure 1: *Impulse responses for IP with p=30 KL functions at different times n.*

If we apply the Fourier transform to every impulse response $\{h[m,n];\ n=0,1,...,N-1\}$ we get the instantaneous frequency responses shown in Figure 2, that are located with respect to a typical normal heartbeat (at top left). Frequency responses are low pass, but with a time-varying response. For ST segment, P and T waves the cut-off frequency is lower than for QRS complex. This behavior is in accordance to the frequency content of each waveform of the ECG signal [6]. We can conclude that the KL orthogonal expansion with IP using p=30 basis functions can represent these frequency components at every time n. This is related to the fact that the KL transform is hand-made from a training set of signals and its first basis functions represent the main signal-morphology.

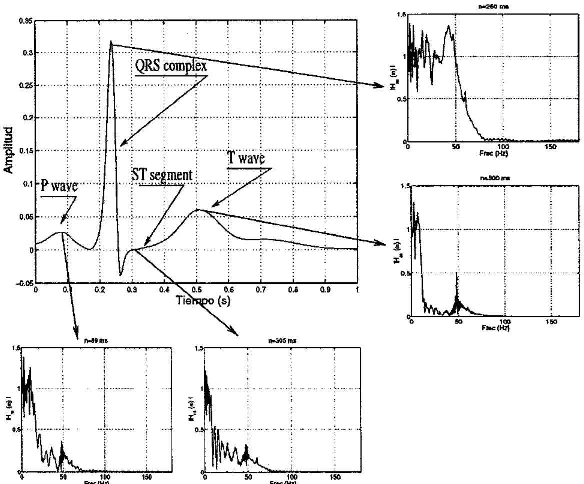

Figure 2: *Frequency responses for IP with p=30 KL functions at different times n.*

Power-line interference of 50 Hz (Europe) and 60 Hz (USA) can be seen in Figure 2 because the signals from the training set were contaminated with these components.

The Fourier transform of the output signal $y[n]$ of the *k-th* occurrence can be related to the input as

$$Y(e^{j\omega}) = \sum_{n=0}^{N-1}\sum_{m=0}^{N-1} x[(k-2)N+m]\, r[m,n]\, e^{-j\omega n}$$

$$= \sum_{m=0}^{N-1} x[(k-2)N+m]\, R(m,e^{j\omega}) \qquad (6)$$

that is, $Y(e^{j\omega})$ is a linear combination of frequency responses where the weights are the input samples and the frequency responses $R(m,e^{j\omega}) = \sum_{n=0}^{N-1} r[m,n]e^{-j\omega n}$ depend on the basis functions used in the expansion. It can be easily demonstrated that $R(m,e^{j\omega}) = e^{-j\omega(N+m)}H^*(e^{-j\omega},m)$ since $r[m,n]=r[n,m]$. The global frequency response of the system can be easily obtained as

$$H_G(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = \frac{\sum_m x[m]\, R(m,e^{j\omega})}{X(e^{j\omega})}. \qquad (7)$$

The filtering effect of the system when the input is misaligned a samples with respect to the basis functions can be studied from (7). In this case, the transfer function will be

$$H_G(e^{j\omega},a) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = \frac{\sum_m x[m-a]\, R(m,e^{j\omega})}{X(e^{j\omega})}. \qquad (8)$$

We show in Figure 3 the distortion introduced in the signal when a misalignment of 52 ms is introduced to a normal heartbeat (record 100 from MIT-BIH database) and p=30 KL basis functions are used. It can be seen that the frequency distortion is higher for misaligned beats than for aligned beats.

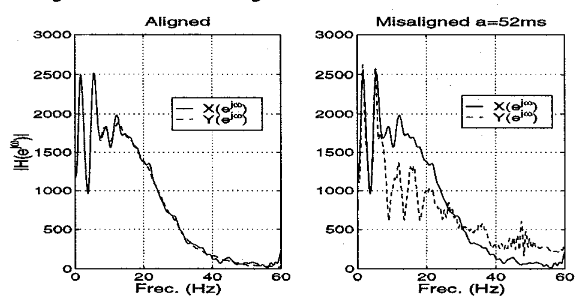

Figure 3: *Frequency distortion due to misaligned signals (left: aligned normal heartbeat; right: 52 ms misaligned heartbeat).*

So, we can conclude that the description of IP as a linear periodic time-variant filter gives a relationship between the input signal at different time instants (around P wave, QRS complex, T wave) with time-variant transfer functions corresponding to those time instants. Also, it allows to interpret the effect of misalignment of the input signal $x[n]$ with respect to the basis functions as a distortion filter.

Other orthogonal transforms whose basis functions have only one frequency component like the Discrete Cosine Transform are not as well suited as KL functions, because the ECG signal has different frequency components at different times within a heartbeat.

3. LMS ALGORITHM

When the input signal is corrupted with uncorrelated noise, adaptive techniques [4, 8] are often used for estimating the orthogonal expansion coefficients. The reference inputs to the adaptive linear combiner shown in Figure 4 are the periodic extension of the basis functions $\widetilde{\Phi}_i[n]$ (deterministic and orthogonal) in contrast to classical situations where random signals are used as reference inputs.

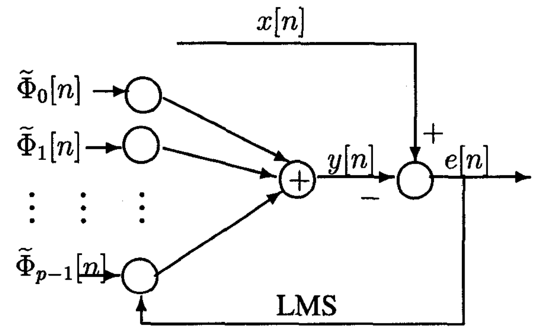

Figure 4: *Adaptive linear combiner for coefficient estimation.*

Several authors have studied the behavior of this system, showing that in the most general case it can be described with a linear time-varying difference equation [2]. Updating the coefficients of the system with $\mathbf{W}[n+1] = \mathbf{W}[n] + 2\mu\, e[n]\widetilde{\mathbf{\Phi}}[n]$ iteratively where $\widetilde{\mathbf{\Phi}}[n]$ is the vector with basis functions values at time n and $\mathbf{W}[n]$ is the coefficient vector at time n and assuming that the weight vector is initialized to the null vector, it is easy to write the difference equation

$$y[n] = 2\,\mu \sum_{m=0}^{n-1} e[m]\,(\widetilde{\Phi}^T[m]\,\widetilde{\Phi}[n]) \qquad (9)$$

that can be re-written as

$$y[n] = 2\mu \sum_{m=0}^{n-1} x[m]\,r[m,n] - 2\mu \sum_{m=0}^{n-1} y[m]\,r[m,n]\,. \qquad (10)$$

This difference equation is recursive, with time increasing order and time-variant coefficients $r[m,n] = \widetilde{\Phi}^T[m]\widetilde{\Phi}[n]$, but with periodic behavior. When all basis functions are used in the expansion $p = N$, the inner products $r[m,n]$ are different from zero only when $m - n = kN$. In this particular case, the estimation system can be described as a linear time-invariant filter, equivalent to a exponential averager with transfer function [2, 3]

$$Y(z) = \frac{2\,\mu\, z^{-N}}{1 + (2\,\mu - 1)\, z^{-N}}\,. \qquad (11)$$

But many applications require a reduced number of functions, such as data compression, parameter extraction for pattern recognition, monitoring, etc.. However, to the best knowledge of the authors, the analysis for a reduced number of coefficients has not been addressed yet.

The recursive equation (10) is difficult to solve directly in order to find the impulse responses that have infinite length because of the recursivity. However, the outputs when the input signals are impulse functions $\delta[n - m]$ can be obtained easily running the filter with this input. Let $r[m,n] = r[m+kN, n+kN]$ be the output at instant n when the input impulse was located at sample m. The impulse responses of the system $\{h[m,n];\ n = 0, 1, \ldots, N-1\}$ can be written as $h[m,n] = r[n-m, n] = h[m, n+kN]$. Different impulse responses of the adaptive system of Figure 4 using p=30 KL functions and a value of μ=0.3 at several times n, are shown in Figure 5. Some differences can be appreciated with respect to the case when all basis functions $p = N$ are used (impulse train with decreasing exponential factor depending on μ).

Applying the Fourier transform to each of the impulse responses we get the instantaneous frequency responses. In Figure 6 a frequency response corresponding to n=250 ms and μ=0.3 is shown.

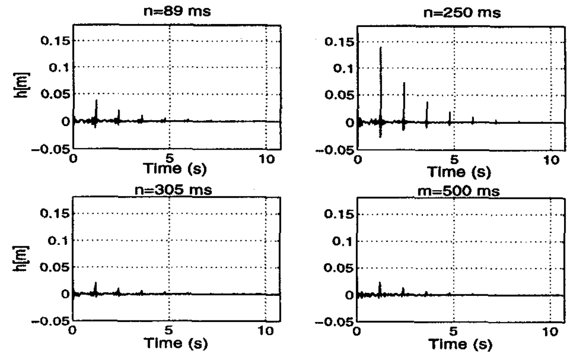

Figure 5: *Impulse responses for LMS $\mu = 0.3$ with p=30 KL functions at different times n.*

The envelope of the frequency responses are very similar (but not identical) to the IP (compare Figure 6 with Figure 2), but frequency responses for LMS are comb filters, so uncorrelated noise (non repetitive components with respect to the heartbeat occurrence time) will be attenuated. This behavior is true for frequencies where the envelope of the IP frequency response is larger than the minimum of adaptive filter lobes, in this case up to around 70 Hz.

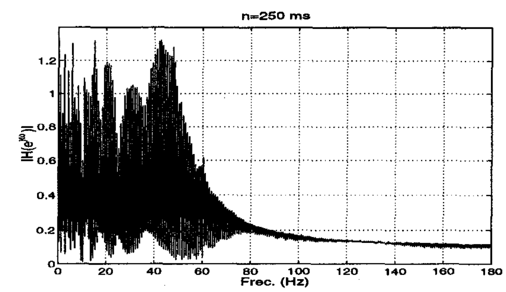

Figure 6: *Frequency responses for LMS $\mu = 0.3$ with p=50 KL functions at time n=250 ms.*

If smaller values of μ are used, the lobes of the frequency response are narrower and closer to the ideal comb filter. To increase this effect smaller values of μ can be selected, increasing the convergence time of the algorithm. This effect is illustrated in Figure 7 where two different values of μ (0.3 and 0.05) are used.

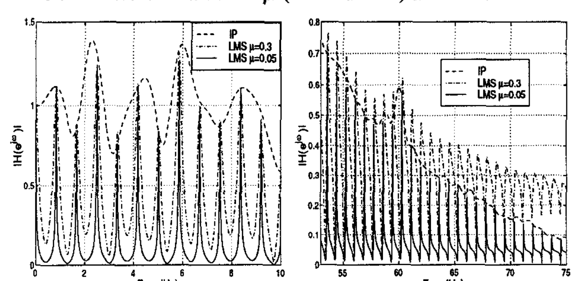

Figure 7: *Low and high details of frequency response at j=250 ms for IP and LMS with $\mu = 0.3$, 0.05 and p=30 KL functions.*

In summary, the adaptive estimation system with LMS and $p < N$ can be described as a linear time-variant periodic filter that can be analyzed by means of instantaneous impulse responses $\{h[m,n];\ n = 0, \cdots, N-1\}$ in a similar way than for IP. The description of the adaptive estimation of orthogonal expansion coefficients using the LMS algorithm is a new way to explain the well-known trade-off for selecting the value of the step factor μ (convergence time and misadjustment reflected in the degradation of the instantaneous frequency responses). Also it explains the combination of both filtering effects: comb filtering due to the adaptive estimation, and low-pass filtering due to the reduced number of functions used in the expansion.

This system can be used to design time-variant filters. From the desired frequency responses (and the envelope of them for LMS case) we would obtain the basis functions $\widetilde{\Phi}[n]$.

4. APPLICATIONS

4.1. ECG data compression

Data compression is one of the most evident applications where the reduced number of functions in orthogonal expansions is a key factor. With the shown description of orthogonal expansion of recurrent signals as a linear time-varying periodic filter we can predict quantitatively which frequency components are well represented at every recurrence time. For example, Figures 2 and 6 illustrate that p=30 KL functions can represent the main frequency components of a heartbeat. Moreover, this description can be a useful tool for testing and comparing behaviors of different orthogonal transforms with variable number of functions.

4.2. Ischemia analysis with the KL transform

Myocardial ischemia is caused by a lack of oxygen over a cardiac area and is reflected on the ECG signal as a low frequency deviations of the ST segment. The Karhunen-Loève has been recently applied to the ST-T complex as a tool for isquemia monitoring [4] showing that is more sensitive than classical local measurements: ST level, position and amplitude of T wave. Using very few KL basis functions it is possible to recover most of the energy of the ST-T complex. Applying the orthogonal expansions description as a linear time-variant filter, we can study the number of basis functions needed for representing the very low frequency components of the ST-T complex.

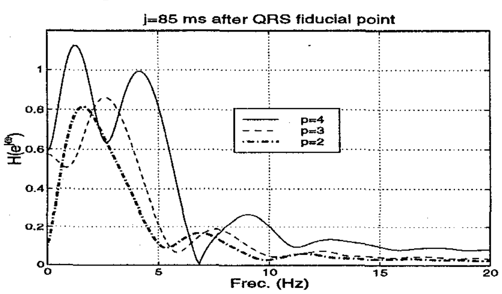

Figure 8: *Frequency responses applying IP with 2, 3 and 4 KL functions for the ST-T complex at n=85 ms after QRS.*

It can be seen from Figure 8 that using only two KL functions the very low frequencies at time n=85 ms after QRS complex are attenuated. Using three or four basis functions improves the representation of the very low frequencies.

5. CONCLUSIONS

In this work two different approaches (inner product and adaptive estimation with the LMS algorithm) are analyzed for estimating the coefficients of orthogonal expansions using a reduced number of functions. We show that both estimation systems are equivalent to a time-variant periodic filter. Inner product has finite impulse responses with duration N samples (length of a occurrence signal), while impulse responses of the LMS are infinite because of the recursive nature of the LMS algorithm. Both systems have the same frequency response envelopes, producing a similar low-pass time varying filtering effect, but with the difference that adaptive frequency responses have comb shape, so they attenuate uncorrelated noise (non-periodic with hearbeat occurrence time).

Using the time-variant periodic filter description of orthogonal expansions shown in this work we can quantitatively know which frequency components and at what time location of recurrent signals are well represented using a reduced number of functions. Therefore it is a useful criteria for determining the number of functions. Applications can be in data compression, parameter extraction for pattern recognition, detection and monitoring, time-variant filter design, etc.. Results are shown for the Karhunen-Loève transform, but can be applied to any orthogonal transform without loss of generality.

6. REFERENCES

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