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# Truncated Orthogonal Expansions of Recurrent Signals: Equivalence to a Linear Time-Variant Periodic Filter

Salvador Olmos, José García, Raimon Jané, and Pablo Laguna

***Abstract*—In this correspondence, we show that orthogonal expansions of recurrent signals like electrocardiograms (ECG's) with a reduced number of coefficients is equivalent to a linear time-variant periodic filter. Instantaneous impulse and frequency responses are analyzed for two classical ways of estimating the expansion coefficients: inner product and adaptive estimation with the LMS algorithm. The obtained description as a linear time-variant periodic filter is a useful tool in order to quantify the distortion produced by the effect of using a reduced number of coefficients in the expansion, and to give frequency criteria to select the appropriate number of functions. Moreover, the misadjustment of the LMS algorithm can be explained as a distortion of the instantaneous frequency response. Experimental results are illustrated with the Karhunen–Loeve transform of ECG signals, but this approach can also be applied to any orthogonal transform.**

***Index Terms*—Data compression, least mean squares methods, transforms.**

## I. INTRODUCTION

Orthogonal expansion is a very well-known technique for signal analysis. It is based on the decomposition of the signal as a linear combination of simple and elementary basis functions [1]. An appropriate choice of the orthogonal basis functions achieves a signal representation, where each coefficient contributes with independent and complementary information, for example, frequency components for the Fourier transform, instantaneous signal values for the identity transform, localized frequency components using the wavelet transform, etc.

When the same number of samples $N$ in the signal under study is used as the number of basis functions in the expansion (exact modeling), the signal energy is completely represented, and the equivalent system can be considered to be the identity function. However, many applications (such as data compression [2]–[5], parameter extraction for pattern recognition, monitoring [6], etc.) require rank reduction. In these applications, the number of basis functions used is reduced to a fraction $p < N$ (undermodeling), and accordingly, some signal components are discarded. The selection of the number of basis functions is the main problem because it is a tradeoff between signal representation capacity and rank reduction.

In this correspondence, we show that the effect of using a reduced number of coefficients in orthogonal expansions of recurrent signals, like electrocardiograms (ECG), can be described as applying a linear time-variant periodic filter to the input signal. This equivalence gives a useful tool in order to quantify *a priori* in the frequency domain the distortion introduced in the reconstructed signal when a variable number of functions is used in the expansion. As a consequence, it can be used to establish a criteria to select the number of basis functions.

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The starting point of this correspondence is the analysis of the LMS algorithm with deterministic inputs [7], where it is proposed as an approximation to the behavior of the LMS algorithm in terms of a linear transfer function. In the more general case, the equivalent system can be described by a recursive finite difference equation with time-varying coefficients. The analysis was illustrated for several examples of deterministic input signals:

- sinusoidal inputs;
- impulse train;
- square waves.

In the particular case of an impulse train, several authors have shown its equivalence to a linear time-invariant comb filter [7]–[10].

In this correspondence, we consider as reference inputs to the LMS algorithm any subset of orthogonal basis functions repeating itself with the periodicity of the recurrent primary signal under analysis. We focus on the effect of using a variable number of basis functions and calculating the equivalent instantaneous impulse and frequency responses. Moreover, the misadjustment of the LMS algorithm can be interpreted by means of the instantaneous frequency responses. An equivalent analysis is also proposed for another classical and well-known technique that is often used to estimate the expansion coefficients: the inner product. Some of the results shown here were presented in [11].

Two classical and well-known techniques are often used to estimate the expansion coefficients: *inner product* (IP) [12], [13] when the value of the signal to noise ratio (SNR) is high, and *adaptive estimation with the LMS algorithm* [14]–[17], when the uncorrelated noise with the signal must be attenuated. Both techniques have been previously analyzed by many authors, but to the best knowledge of the authors, the analysis with a variable number of basis functions has yet to be addressed. The aim of this correspondence is to give a new interpretation of both techniques as a linear time-variant periodic filter. In order to illustrate the equivalence, instantaneous impulse and frequency responses of the system are calculated when the Karhunen–Loeve (KL) transform [2], [13] is applied to ECG signals. The usefulness of the new interpretation is shown in two applications: the time-varying distortion effect of the number of basis functions used for data compression of ECG signal with the KL transform and the filtering effect when compacting information for ischemia analysis of the ECG signal.

## II. Inner Product

The inner product (IP) is the most common way to estimate the orthogonal expansion coefficients of signals with high values of SNR. The IP is the solution to the problem of minimizing the mean square error between the original signal and a reduced linear combination of basis functions [13]. In addition, the IP has a geometric interpretation as the orthogonal projection of the signal vector onto the signal subspace spanned by the basis functions used in the expansion.

The reconstructed signal $y[n]$ for the $k$th occurrence of a recurrent $N$-length input signal $x[n]$, using a reduced number $p < N$ of coefficients, can be written as

$$y[(k-1)N+n] = \sum_{i=0}^{p-1} c_i^{k-1}\tilde{\Phi}_i[n], \qquad 0 < n < N-1 \tag{1}$$

where $\tilde{\Phi}_i[n]$ are the periodic extension of the basis functions $\Phi_i[n]$, and $c_i^{k-1}$ are the coefficients of the inner product between the basis functions and the signal from the previous occurrence (applying causality to the filter)

$$c_i^{k-1} = \sum_{m=0}^{N-1} \tilde{\Phi}_i[m]x[(k-2)N+m]; \qquad 0 < i < p-1. \tag{2}$$

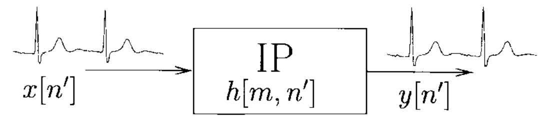



Fig. 1. Block diagram of the inner product as a linear time-variant periodic filter.

The input–output equation is obtained by substituting (2) in (1)

$$y[(k-1)N+n] = \sum_{m=0}^{N-1} x[(k-2)N+m]r[m, n] \qquad 0 < n < N-1 \tag{3}$$

where $r[m, n] = \sum_{i=0}^{p-1} \tilde{\Phi}_i[m]\tilde{\Phi}_i[n] = \tilde{\mathbf{\Phi}}^T[m]\tilde{\mathbf{\Phi}}[n]$ is the inner product between the basis functions vectors at time instants $m$ and $n$, and $\tilde{\mathbf{\Phi}}[n]$ is the basis functions vector at time $n$. The output at occurrence instant $n$ can be seen as a linear combination of input samples from the last occurrence ($N$ samples delay) with time-varying values of the basis functions inner product $r[m, n]$; that is, IP can be described as a linear time-variant filter that is also periodic because $r[m, n] = r[m+N, n] = r[m, n+N]$. In order to find the instantaneous impulse responses, (3) should be written as a linear convolution of the input signal $x[n]$, where $n' = (k-1)N+n$ denotes the time index

$$y[n'] = \sum_{m=-\infty}^{\infty} x[m]h[n'-m, n'] \tag{4}$$

with the $N$ samples finite duration impulse responses $\{h[m, n];\ n = 0, 1, \cdots, N-1\}$

$$h[m, n] = \begin{cases} r[n-m+N, n], & \text{for } m = n+1, n+2, \cdots, n+N \\ 0, & \text{elsewhere.} \end{cases} \tag{5}$$

The first index $m$ of $h[m, n]$ denotes the index for the impulse response shape, and the second one $n$ denotes the time instant when the impulse response is valid. Fig. 1 illustrates the interpretation of the inner product as a linear time-variant periodic filter that is applied to the input signal occurrences.

The same result can also be easily obtained using the vector notation, where the reconstructed occurrence signal vector $\mathbf{Y}$ can be obtained as the matrix product

$$\mathbf{Y} = \mathbf{T}^H\mathbf{TX} = \mathbf{HX} \tag{6}$$

where $\mathbf{T}$ is the $p \times N$ matrix whose columns are the basis functions vectors used in the expansion $\mathbf{T} = [\mathbf{\Phi}[0]\mathbf{\Phi}[1]\cdots\mathbf{\Phi}[N-1]]$, and $\mathbf{X}$ is the input signal occurrence vector. Several digital signal processing textbooks state the equivalence between matrix product and linear convolution operators [3], [18]. In a general case (for noncomplete orthogonal expansions), the $N \times N$ matrix $\mathbf{H}$ will not be Toeplitz, and accordingly, the output can be calculated as the linear convolution of the input with a finite length time-variant impulse response, whose values are determined by the inner product values of the basis functions.

In order to quantify this filtering effect, instantaneous impulse and frequency responses are calculated, depending on the number $p$ of basis functions used in the expansion. The KL transform [2], [13] of ECG signals is used as an example, although other orthogonal expansions can be considered without loss of generality. The basis functions are estimated [19] from a training set of ECG signals of MIT-BIH Arrythmia [20] and ESC-STT [21] databases (resampled to 360 Hz). In Fig. 2, impulse responses $h[m, n]$ are shown for different values of $n$, when $p = 30$ KL basis functions are used

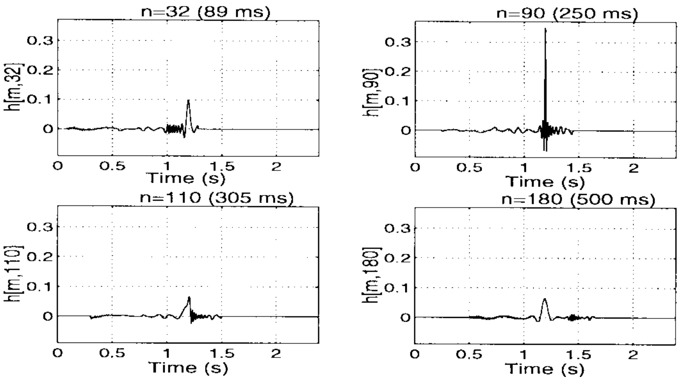



Fig. 2. Impulse responses for IP with $p = 30$ KL functions at different times $n$.

in the expansion. The impulse responses are of finite length with duration $N = 430$ samples (heartbeat length).

If we apply the Fourier transform to each impulse response $\{h[m, n];\ n = 0, 1, \cdots, N-1\}$, we get the instantaneous frequency responses $H(e^{j\omega}, n)$, which are shown in Fig. 3, located with respect to a typical normal heartbeat (at top left). Frequency responses are lowpass but with a time-varying response. For the ST segment and P and T waves, the cutoff frequency is lower than for the QRS complex. This behavior is in accordance with the frequency content of each waveform of the ECG signal [22]. We can observe that the KL orthogonal expansion with the IP using $p = 30$ basis functions can represent the main frequency components of each ECG waveform at every time $n$. This behavior is related to the fact that the KL transform is handmade from a training set of ECG signals, and its first basis functions represent the main signal-morphology.

Other orthogonal transforms whose basis functions have only one frequency component throughout time, like the discrete cosine transform, are not as well suited as KL functions to represent the ECG with a reduced number of coefficients, because the ECG signal has different frequency components at different times within a heartbeat.

The Fourier transform of the output signal $y[n]$ at the $k$th occurrence can be related to the input as

$$\begin{aligned} Y(e^{j\omega}) &= \sum_{n=0}^{N-1} y[(k-1)N + n]e^{-j\omega n} \\ &= \sum_{n=0}^{N-1}\sum_{m=0}^{N-1} x[(k-2)N+m]r[m, n]e^{-j\omega n} \\ &= \sum_{m=0}^{N-1} x[(k-2)N+m]R(m, e^{j\omega}) \end{aligned} \tag{7}$$

that is, $Y(e^{j\omega})$ is a linear combination of frequency responses where the weights are the input samples of the previous signal occurrence, and the frequency responses depend on the basis functions used in the expansion $R(m, e^{j\omega}) = \sum_{n=0}^{N-1} r[m, n]e^{-j\omega n}$. Using (5) and the fact that $r[m, n] = r[n, m]$, it can be easily demonstrated that

$$R(m, e^{j\omega}) = e^{-j\omega(m+N)}H^*(e^{j\omega}, m) \tag{8}$$

where $H(e^{j\omega}, m)$ is the frequency response of the system at instant $m$ (Fourier transform of impulse response $h[n, m]$). In a similar way, the instantaneous frequency response $H(e^{j\omega}, n) = \sum_{m=-\infty}^{\infty} h[m, n]e^{-j\omega n}$ can be calculated as

$$H(e^{j\omega}, n) = e^{-j\omega(n+N)}R^*(e^{j\omega}, n). \tag{9}$$

Thus, the instantaneous frequency responses $H(e^{j\omega}, n)$ depend only on the basis functions used in an easy way. The global frequency response of the system can be straightforwardly obtained as

$$H_G(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = \frac{\sum_m x[m]R(m, e^{j\omega})}{X(e^{j\omega})}. \tag{10}$$

The global frequency response $H_G(e^{j\omega})$ relates the original signal to the reconstructed by means of the basis representation $R(m, e^{j\omega})$.

Many biomedical signals are time locked to a stimulus that can be internal like electrocardiograms or can be external, as in the case of evoked potentials [8]. In the case of ECG, the time-reference point is defined from identifying a signal waveform, such as the QRS complex. In order to apply orthogonal expansions to ECG signals, we need to estimate the time-reference point of the signal, using QRS detectors [23], [24]. Timing errors can appear in the estimation process due to the noise presence. Therefore, it can be interesting to analyze the behavior of the inner product estimation of the signal when a misalignment of $a$ samples occurs between the signal occurrence and the basis functions. In this case, the global frequency response will be

$$H_G(e^{j\omega}, a) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = \frac{\sum_m x[m-a]R(m, e^{j\omega})}{X(e^{j\omega})}. \tag{11}$$

We show in Fig. 4 the distortion introduced in the output signal when a misalignment of 52 ms is introduced to a normal heartbeat (record 100 from MIT-BIH database), and $p = 30$ KL basis functions are used. Comparing the overprinted original signal $X(e^{j\omega})$ with the reconstructed one $Y(e^{j\omega})$, it can be seen that the frequency distortion is higher for misaligned beats than for aligned beats.

We can conclude that the description of IP as a linear time-variant periodic filter gives a relationship between the input signal at different time instants (around the P wave, the QRS complex, and the T wave) with time-variant transfer functions corresponding to those time instants. In addition, it allows the interpretation of the effect of misalignment of the input signal $x[n]$ with respect to the basis functions as a distortion filter.

The description shown here could also be used to design linear time-variant periodic filters. The problem would be addressed in the reverse direction: From desired instantaneous frequency responses $H(e^{j\omega}, n)$ and using (9), we could obtain the basis functions products $r[m, n] = \tilde{\mathbf{\Phi}}^T[m]\tilde{\mathbf{\Phi}}[n]$. From these values, we should obtain the values of the basis functions $\tilde{\mathbf{\Phi}}[n]$.

## III. Adaptive Estimation with the LMS Algorithm

When the input signal is corrupted with uncorrelated noise, adaptive techniques [14]–[17] are often used to estimate the orthogonal expansion coefficients. The reference inputs to the adaptive linear combiner [15] shown in Fig. 5 are the periodic extension of the basis functions $\tilde{\Phi}_i[n]$ (deterministic and orthogonal) in contrast with classical adaptive schemes where two random signals are used as reference inputs. The primary input $x[n]$ is the noisy recurrent signal under study.

The weight vector $\mathbf{W}[n]$ is updated with the LMS equation

$$\mathbf{W}[n+1] = \mathbf{W}[n] + 2\mu e[n]\tilde{\mathbf{\Phi}}[n] \tag{12}$$

where $\tilde{\mathbf{\Phi}}[n]$ is the vector of basis functions values at time $n$. Assuming that the weight vector is initialized to the null vector $\mathbf{W}[0] = \mathbf{0}$, Clarkson [7], [9] analyzed the behavior of this system, showing that in the most general case, it can be described as a finite difference equation

$$y[n] = 2\mu\sum_{m=0}^{n-1} e[m](\tilde{\mathbf{\Phi}}^T[m]\tilde{\mathbf{\Phi}}[n]) \tag{13}$$

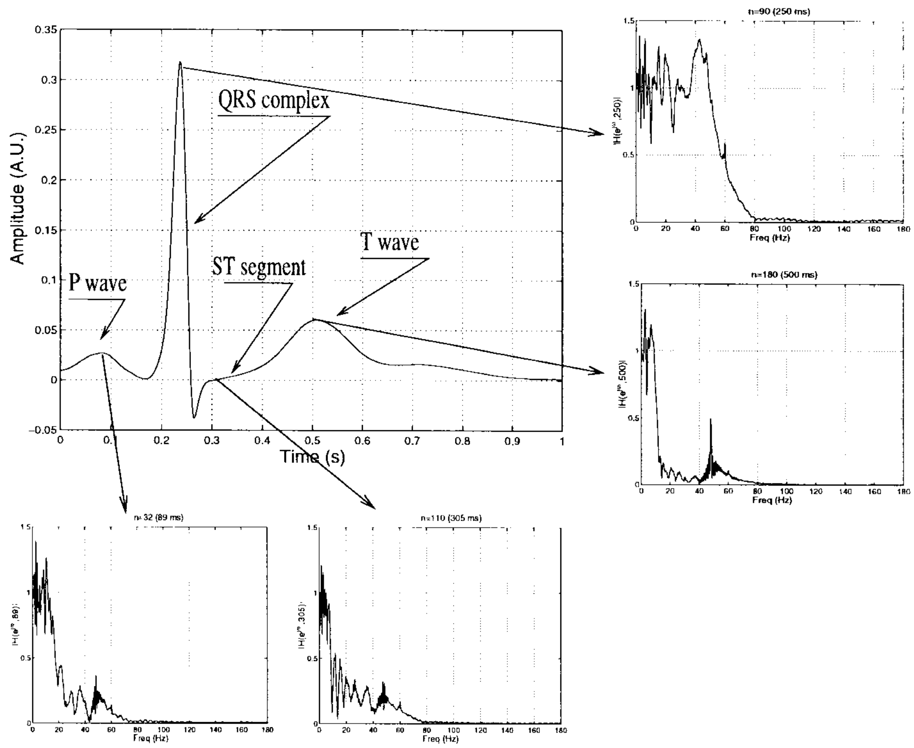



Fig. 3. Frequency responses $H(e^{j\omega}, n)$ for IP with $p = 30$ KL functions at different times $n$.

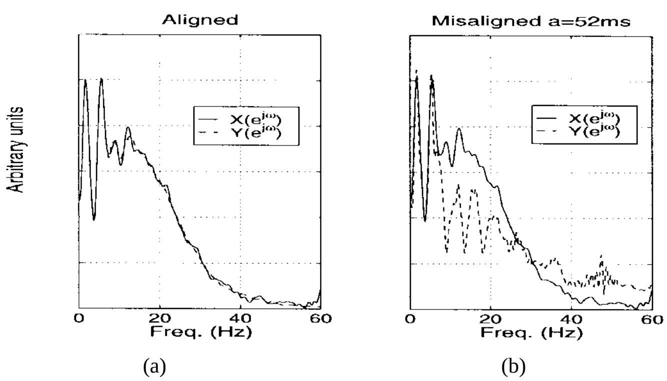



Fig. 4. Frequency distortion due to misaligned occurrences. (a) Aligned normal heartbeat. (b) 52-ms misaligned heartbeat).

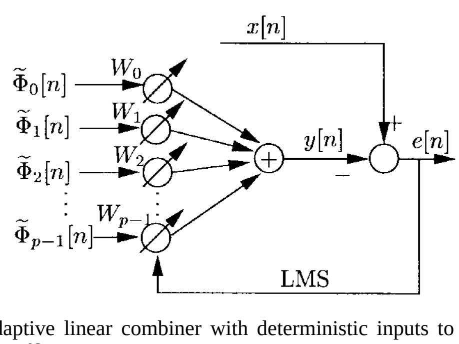



Fig. 5. Adaptive linear combiner with deterministic inputs to estimate the expansion coefficients.

that can be rewritten as

$$y[n] = 2\mu \sum_{m=0}^{n-1} x[m]r[m, n] - 2\mu \sum_{m=0}^{n-1} y[m]r[m, n]. \qquad (14)$$

This finite difference equation is recursive, with time increasing order and time-variant periodic coefficients $r[m, n] = \tilde{\mathbf{\Phi}}^T[m]\tilde{\mathbf{\Phi}}[n]$.

When all basis functions are used in the expansion $p = N$ (exact modeling), the inner product values are

$$r[m, n] = \begin{cases} 1, & \text{for } m - n = kN \\ 0, & \text{otherwise.} \end{cases} \qquad (15)$$

In this particular case, the finite difference equation (14) defines a linear time-invariant filter with transfer function

$$H(z) = \frac{Y(z)}{X(z)} = \frac{2\mu z^{-N}}{1 + (2\mu - 1)z^{-N}}. \qquad (16)$$

Several authors had previously analyzed the adaptive linear combiner system when the reference input signals are the complete set of impulse functions (equivalent to the identity transform) obtaining the same transfer function [7], [8], [10]. The same result is now generalized for any complete set of orthogonal basis functions. Its frequency response $H(e^{j\omega})$ is a comb filter [10] whose lobes repeat at frequencies that are multiples of the normalized fundamental frequency $\omega_0 = (2\pi/N)$. The $-3$ dB cut-off frequency of each lobe is $(\mu/\pi)\omega_0$ far from the central frequency of the lobe. High values of the step-size $\mu$ imply simultaneously wider lobes of the frequency response and higher values of the misadjustment. In [10], the steady-state *excess* MSE $\xi^{ex}[\infty]$ was interpreted as the residual noise that passes through the lobes spread, and now, it can be easily calculated as

$$\xi^{ex}[\infty] = \frac{1}{2\pi} \int_{-\pi}^{\pi} |H(e^{j\omega})|^2 S_n(e^{j\omega})\, d\omega \qquad (17)$$

where $S_n(e^{j\omega})$ is the power spectral density of the noise. In the particular case of white noise $S_n(e^{j\omega}) = \sigma^2$, the integral (17) can be easily calculated using the Parseval relation as

$$\xi^{ex}[\infty] = \sigma^2 \sum_{n=-\infty}^{\infty} |h[n]|^2 = \sigma^2 \frac{\mu}{1-\mu} \qquad (18)$$

giving the same result of *excess* MSE previously obtained by several authors [8], [10]. In the case of colored noise, we can obtain the same

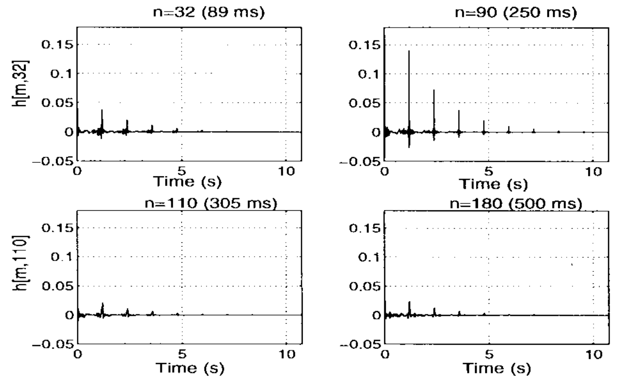



Fig. 6. Impulse responses $h[m, n]$ for LMS $\mu = 0.3$ with $p = 30$ KL functions at different times $n$.

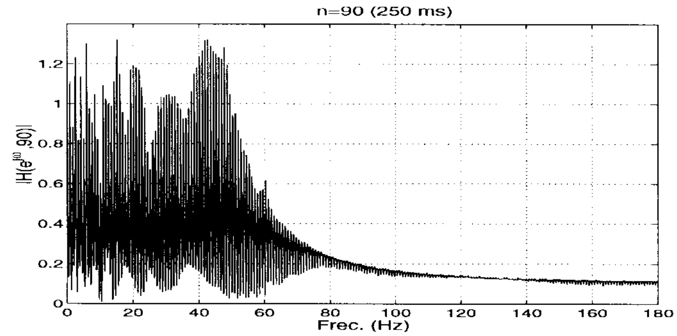



Fig. 7. Frequency response $H(e^{j\omega}, n)$ for LMS with $\mu = 0.3$ and $p = 30$ KL basis functions at time $n = 90$ (250 ms).

result applying the spectral factorization. The time-invariant impulse response is a causal impulse train with exponentially decreasing amplitude that can be expressed as

$$h[n] = 2\mu \sum_{k=1}^{\infty} (1 - 2\mu)^{k-1} \delta[n - kN]. \tag{19}$$

Several authors have shown the equivalence of this filter with an exponential weighted averager [8].

However, many applications require a reduced number of functions, such as data compression, parameter extraction for pattern recognition, monitoring, etc. To our best knowledge, the analysis for a reduced number of coefficients (undermodeling) has not yet been addressed.

The recursive equation (14) is difficult to solve directly in order to find the impulse responses that have infinite length because of the recursivity. However, the outputs when the input signals are impulse functions $\delta[n - a]$ located at sample $a$ can be easily obtained running the filter with this input. Let $f[a, n]$ be the output at instant $n$ when the input impulse was located at sample $a$. The impulse responses of the system $\{h[m, n]; n = 0, 1, \cdots, N - 1\}$ can be written as $h[m, n] = f[n - m, n]$ because $f[a, n]$ can be expressed as the linear convolution

$$\begin{aligned} f[a, n] &= \sum_{m=-\infty}^{\infty} h[m, n] x[n - m] \\ &= \sum_{m=-\infty}^{\infty} h[m, n] \delta[n - a - m] = h[n - a, n]. \end{aligned} \tag{20}$$

Different impulse responses $h[m, n]$ of the adaptive system using $p = 30$ KL basis functions and a value of the step-size $\mu = 0.3$ at different times $n$ are shown in Fig. 6. Some differences can be appreciated with respect to the case when all basis functions $p = N$ are used [impulse train with exponential decreasing factor depending on the step-size $\mu$ at (19)]. In truncated orthogonal expansions, there is an interaction between the basis functions due to the noncompleteness of the subspace (the inner product values $r[m, n]$ are different from zero), and this is illustrated in the burst-like decaying amplitude episodes and not only impulses.

Applying the Fourier transform to each of the impulse responses $h[m, n]$, we get the corresponding instantaneous frequency responses $H(e^{j\omega}, n)$. Fig. 7 illustrates the frequency response at the time instant $n = 90$ (250 ms) near the peak of the QRS complex. Frequency responses for the LMS algorithm appear to be comb filters; therefore, uncorrelated noise (nonrepetitive components with respect to the heartbeat occurrence time) will be attenuated [8], [10].

If smaller values of the step-size $\mu$ are used, the lobes of the frequency response are narrower and closer to the ideal comb filter. This effect is illustrated in Fig. 8, where details of low and medium frequencies of the same frequency response shown at Fig. 7 can be seen for two different values of the step-size $\mu$ (0.3 and 0.05). The inner product frequency response at the same time instant $n$ is also shown for comparison. The envelope of the frequency response for the LMS algorithm is very similar (but not identical) to the IP (compare Fig. 7 with Fig. 3).

From Fig. 8, we can see that frequency response envelopes of the LMS algorithm with small values of the step-size $\mu$ are a better approximation to the inner product frequency response than with high values of $\mu$. This effect can be explained analyzing the LMS update equation with very small values of the step-size $\mu$. We will calculate the output at sample $n$ of the adaptive filter to an impulse located at sample $m_0$ $f[m_0, n] = y[n] = \mathbf{W}^T[n]\tilde{\mathbf{\Phi}}[n]$, applying the recursive equation (12) to the weight vector $\mathbf{W}[n]$. When the input signal is an impulse function $x[n] = \delta[n - m_0]$, the error signal $e[n]$ at time instants $n > m_0$ is directly $e[n] = -y[n]$, and the weight vector update equation can be written as

$$\mathbf{W}[n+1] = (\mathbf{I} - 2\mu\tilde{\mathbf{\Phi}}[n]\tilde{\mathbf{\Phi}}^T[n])\mathbf{W}[n] \qquad n \geq m_0. \tag{21}$$

If we apply (21) iteratively, we have

$$\mathbf{W}[n] = \begin{cases} 0 & n \leq m_0 \\ 2\mu\tilde{\mathbf{\Phi}}[m_0] & n = m_0 + 1 \\ \mathbf{F}_{n-1, m_0+1}\mathbf{W}[m_0 + 1] & n \geq m_0 + 2 \end{cases} \tag{22}$$

where $\mathbf{F}_{i, j}$ represents the product of the time-variant transition matrices

$$\mathbf{F}_{i, j} = \begin{cases} (\mathbf{I} - 2\mu\tilde{\mathbf{\Phi}}[i]\tilde{\mathbf{\Phi}}^T[i])(\mathbf{I} - 2\mu\tilde{\mathbf{\Phi}}[i-1]\tilde{\mathbf{\Phi}}^T[i-1]) & \\ \quad \cdots (\mathbf{I} - 2\mu\tilde{\mathbf{\Phi}}[j]\tilde{\mathbf{\Phi}}^T[j]), & i > j \\ \mathbf{I} - 2\mu\tilde{\mathbf{\Phi}}[i]\tilde{\mathbf{\Phi}}^T[i], & i = j. \end{cases} \tag{23}$$

Making use of the periodic behavior of the basis functions $\tilde{\mathbf{\Phi}}[n]$, we can write a recursive relation of the weight vector every $N$ samples as

$$\mathbf{W}[n + N] = \mathbf{F}_{n+N, n+1}\mathbf{W}[n]. \tag{24}$$

The product of the transition matrices over one complete signal occurrence $\mathbf{F}_{n+N, n+1}$ will be a $N$-degree polynomial of $\mu$ that can be written as

$$\mathbf{F}_{n+N, n+1} = \mathbf{I} - 2\mu\mathbf{A}_1 + (2\mu)^2\mathbf{A}_2 - (2\mu)^3\mathbf{A}_3 + \cdots \tag{25}$$

where $\mathbf{A}_1 = \sum_{k=n+1}^{n+N} \tilde{\mathbf{\Phi}}[k]\tilde{\mathbf{\Phi}}^T[k] = \mathbf{I}$. If small values of $\mu$ are used ($\mu \ll 1$), the last equation can be approximated to $\mathbf{F}_{n+N, n+1} \simeq (1 - 2\mu)\mathbf{I}$. If we apply iteratively the update weight vector equation (24), we get

$$\mathbf{W}[n + kN] \simeq (1 - 2\mu)^k\mathbf{W}[n] \tag{26}$$

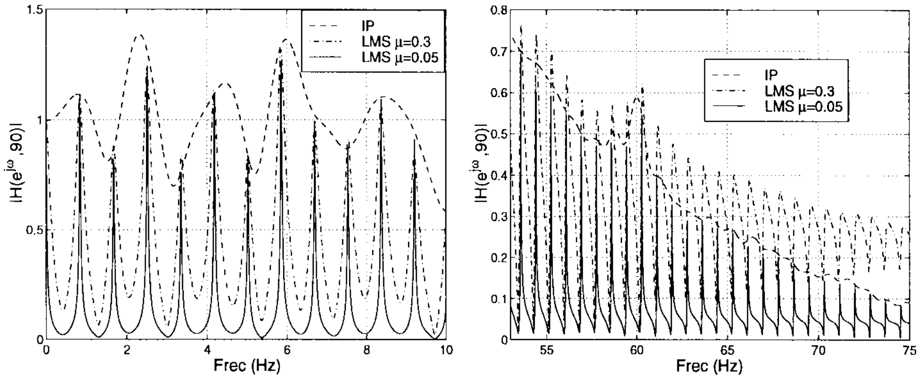



Fig. 8. Low- and high-frequency details of the frequency response $H(e^{j\omega}, n)$ at time $n = 90$ (250 ms) for IP and LMS with $p = 30$ KL basis functions and $\mu = 0.3$ and 0.05.

which means the weight vector trajectory along occurrences keeps the same shape except for an exponentially decreasing factor. As a consequence, the output signal will be approximately equal to a unique waveform with a constant decreasing factor $(1-2\mu)$ at every signal occurrence

$$y[n+kN] \simeq (1-2\mu)^k y[n]. \tag{27}$$

Finally, the output of the filter to an impulse function located at sample $m_0$ for small values of $\mu$ ($\mu^2$ is neglected) can be calculated using (22) and (27)

$$f[m_0, n] \simeq \begin{cases} 0, & n \le m_0 \\ 2\mu(1-2\mu)^k r[m_0, n], & n > m_0; k = \lfloor \frac{n}{N} \rfloor \end{cases}. \tag{28}$$

This equation can be interpreted as the convolution of a causal impulse train with decreasing amplitude [the same impulse response for the case of complete expansion case of (19)] and the values of the basis functions inner products $r[m, n]$. The Fourier transform of $f[m_0, n]$ will be

$$\begin{aligned} F(m_0, e^{j\omega}) &= \sum_{n=-\infty}^{\infty} f[m_0, n] e^{-j\omega n} \\ &= \sum_{n=m_0+1}^{\infty} f[m_0, n] e^{-j\omega n}. \end{aligned} \tag{29}$$

Decomposing the series in occurrence intervals and applying (28), we get

$$\begin{aligned} &F(m_0, e^{j\omega}) \\ &\quad = \sum_{k=0}^{\infty} 2\mu(1-2\mu)^k \sum_{q=m_0+1}^{m_0+N} r[m_0, q] e^{-j\omega(q+kN)} \end{aligned} \tag{30}$$

which can be expressed as

$$\begin{aligned} F(m_0, e^{j\omega}) &= \left( \sum_{k=0}^{\infty} 2\mu(1-2\mu)^k e^{-j\omega kN} \right) \\ &\quad \cdot \left( \sum_{q=m_0+1}^{m_0+N} r[m_0, q] e^{-j\omega q} \right) \\ &= \frac{2\mu}{1+(2\mu-1)e^{-j\omega N}} R(m_0, e^{j\omega}). \end{aligned} \tag{31}$$

Remembering that the frequency response of the time-invariant system (comb filter) corresponding to the adaptive system with complete expansions is written in (16), and using (9), we can write

$$\begin{aligned} F(m_0, e^{j\omega}) &= \frac{2\mu e^{-j\omega N}}{1+(2\mu-1)e^{-j\omega N}} e^{j\omega N} R(m_0, e^{j\omega}) \\ &= e^{-j\omega m_0} H_{\text{LMS-CE}}(e^{j\omega}) H_{\text{IP}}^*(e^{j\omega}, m_0) \end{aligned} \tag{32}$$

where $H_{\text{IP}}^*(e^{j\omega}, m_0)$ is the IP frequency response at instant $m_0$, and $H_{\text{LMS-CE}}(e^{j\omega})$ is the time-invariant LMS frequency response with complete expansions. The exponential factor $e^{-j\omega m_0}$ is generated by the delay of the input signal $\delta[n-m_0]$. Thus, we have demonstrated that the Fourier transform of the output $f[m_0, n]$ for low values of the step-size $\mu$ is equivalent to the product of both frequency responses: comb filter and inner product.

A similar procedure could be applied to the instantaneous frequency response of the LMS algorithm $H_{\text{LMS}}(e^{j\omega}, m_0)$ using (20)

$$\begin{aligned} H_{\text{LMS}}(e^{j\omega}, m_0) &= \sum_{m=-\infty}^{\infty} h[m, m_0] e^{-j\omega m} \\ &= \sum_{m=\infty}^{\infty} f[m_0 - m, m_0] e^{-j\omega m}. \end{aligned} \tag{33}$$

Following the same argument as before

$$\begin{aligned} &H_{\text{LMS}}(e^{j\omega}, m_0) \\ &\quad = \sum_{m=1}^{\infty} f[m_0 - m, m_0] e^{-j\omega m} \\ &\quad = \left( \sum_{k=0}^{\infty} 2\mu(1-2\mu)^k e^{-j\omega kN} \right) \\ &\qquad \cdot \left( \sum_{q=m_0+1}^{m_0+N} r[m_0 - q, m_0] e^{-j\omega q} \right) \\ &\quad = \frac{2\mu}{1+(2\mu-1)e^{-j\omega N}} e^{-j\omega(m_0+N)} R^*(e^{j\omega}, m_0) \\ &\quad = e^{-j\omega m_0} H_{\text{LMS-CE}}(e^{j\omega}) H_{\text{IP}}(e^{j\omega}, m_0). \end{aligned} \tag{34}$$

The instantaneous frequency responses of the LMS algorithm, when low values of the step-size are used, is equal to the product of both frequency responses: the comb filter corresponding to the adaptive estimation system with complete expansions and the inner product frequency response due to the effect of using a reduced number $p$ of coefficients in the expansion. This result was previously observed at Fig. 8.

In summary, the adaptive estimation system with the LMS algorithm with $p < N$ can be described as a linear time-variant periodic filter that can be analyzed by means of instantaneous impulse responses $\{h[m, n];\ n = 0, \cdots, N-1\}$ in a similar way than for IP. This description of the adaptive estimation of orthogonal expansion coefficients using the LMS algorithm explains the combination of both filtering effects: comb filtering due to the adaptive estimation and lowpass filtering due to the reduced number of functions used

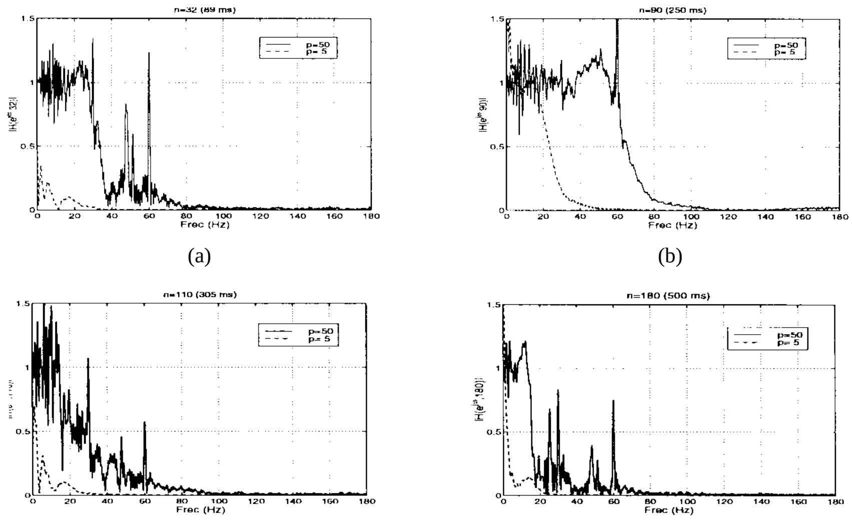



Fig. 9. Frequency responses $H(e^{j\omega}, n)$ for IP with $p = 5$ and 50 KL functions at different times $n$. (a) P wave instant $n = 32$ (89 ms). (b) QRS complex instant $n = 90$ (250 ms). (c) ST segment instant $n = 110$ (305 ms). (d) T wave instant $n = 180$ (500 ms).

in the expansion. Furthermore, it is a new way to explain the well-known tradeoff for selecting the value of the step factor $\mu$ (convergence time and misadjustment reflected in the degradation of the instantaneous frequency responses). Low values of the step-size $\mu$ are recommended for estimating signals embedded in noise because the misadjustment is larger for higher values of the step-size $\mu$ [9], [14], [15]. With the description shown here, we observe that the instantaneous frequency responses for low values of the step-size $\mu$ can be approximated as the product of the comb filter [8] and the inner product frequency response. Low $\mu$ values imply thinner lobes in the frequency response, and thus, the noise power that passes through the filter is reduced. However, low values of the step-size also increase the convergence time. Higher values of the step-size generate two effects.

- The lobes of the comb filter (16) are wider.
- The lowpass frequency response is distorted.

In Fig. 8, we can see that this distortion (bias from the product of comb filter and inner product frequency response) is larger at high frequencies. Therefore, this is a new way to interpret and quantify the misadjustment of the LMS algorithm.

## IV. Applications

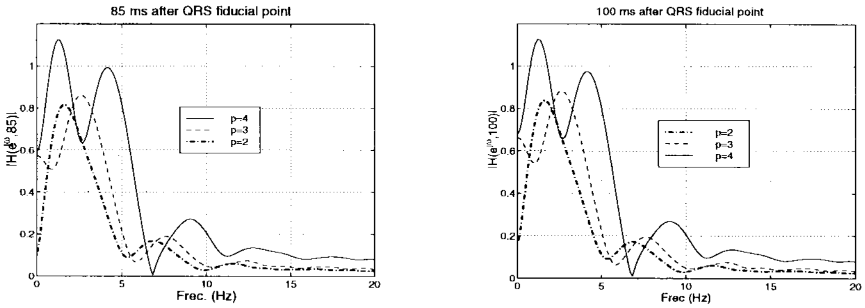
### A. ECG Data Compression

Data compression is one of the most evident applications where the reduced number of functions in orthogonal expansions is a key factor [2]. ECG data compression systems based on orthogonal expansions, like [17], [19], and [25], get better rate-distortion tradeoff than methods based on interpolation techniques [4]. With the shown description of orthogonal expansion of recurrent signals as a linear time-variant periodic filter, we can quantitatively predict which frequency components are well represented at every occurrence time using a variable number of basis functions. For example, Figs. 3 and 7 illustrate that $p = 30$ KL basis functions can represent the main frequency components of a heartbeat. Moreover, this description can be a useful tool for testing and comparing behaviors of different orthogonal transforms with variable number of basis functions.

For example, low bit rate data compression systems use a very low number of basis functions to represent the ECG signal. As a consequence, many frequency components will be attenuated. In Fig. 9, we show the frequency responses (dash lines) at several time instants when only $p = 5$ KL basis functions are used in the expansion and the coefficients are estimated with IP. We can see that the main frequency components of QRS complex [22] [Fig. 9(b)] are well represented, whereas other waveforms like the P wave and the ST segment [Fig. 9(a) and (c), respectively] have a large attenuation. This result is in accordance to the fact that the KL transform basis functions were sorted in decreasing value of the covariance matrix eigenvalues. Therefore, the first basis functions represent the heartbeat intervals corresponding to the highest signal energy. As the QRS complex and T wave have more energy than the P wave and the ST segment, they are better represented when few functions are used. Sometimes, however, the clinical information underlying in latter intervals is very useful for diagnostic purposes. As a consequence, higher values of the number of functions $p$ should be used in order to avoid distortion at the P wave and ST segment.

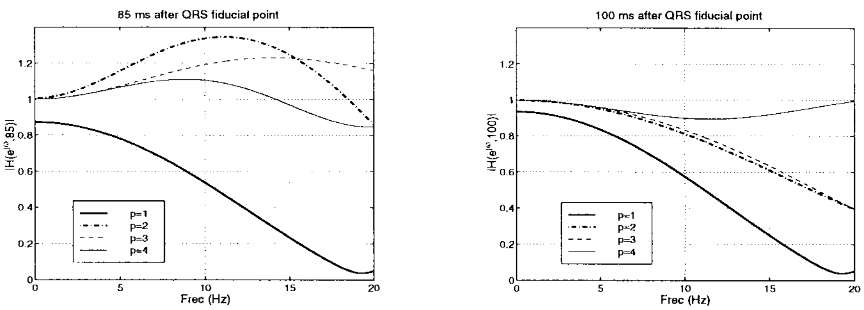
In contrast, when the number of basis functions is very high, the signal (as well as noise) representation capacity increases. For example, in Fig. 9, we also show several instantaneous frequency responses when $p = 50$ KL basis functions (solid lines). Power-line interference of 50 Hz (Europe) and 60 Hz (USA) can be seen because the signals from the training set were contaminated with these components. Records from databases were recorded on analog tapes with battery-powered recorders, so most of the 50/60-Hz noise present in the database arose during playback. Several records were digitized at twice real time; therefore, this noise appears at 25/30 Hz (and multiples) relative to real time.

In this correspondence, we have shown the equivalence of truncated orthogonal expansions with a linear time-variant filter. However, we have not addressed an automatic method to estimate the appropriate number of basis functions $p$ at any application. We believe that this is an extension to the presented work that should consider the particular interest of each application. For example, for data compression of ECG signals, the selection of the number $p$ of basis functions could be done by first defining the desired minimum cut-off frequencies at different time instants of the signal occurrence (P wave, QRS complex, ST segment, T wave) according to the clinical information of the ECG waveforms and, second, calculating the instantaneous frequency responses for a variable and growing value of $p$ until the required specifications are obtained. At the end




(a) (b)

Fig. 10. Frequency responses applying IP with two, three, and four KL basis functions for the ST-T complex at two different time instants of the ST segment. (a) Time instant 85 ms after QRS mark. (b) Time instant 100 ms after QRS mark.




(a) (b)

Fig. 11. Frequency responses applying IP with two, three, and four KL functions for the ST segment at different time instants of the ST segment. (a) Time instant 85 ms after QRS mark. (b) Time instant 100 ms after QRS mark.

of the process, we obtain the minimum value of $p$ that can represent all clinical information of the signal.

### *B. Ischemia Analysis with the KL Transform*

Myocardial ischemia is caused by a lack of blood supply over a cardiac area and is reflected on the ECG signal as a low-frequency deviation of the ST segment. The KL transform has been recently applied to the ST-T complex (composed by the ST segment and the T wave) as a tool for the monitoring of ischemia [6]. The KL sensitivity to detect ischemic changes in the repolarization induced by angioplasty was larger (85%) than that obtained by traditional indexes based on local measurements on the ECG: ST level (64%), T wave amplitude (37%), or T wave position (33%) [26].

Applying the orthogonal expansions description as a linear time-variant periodic filter, we can study the number of basis functions needed for representing the very low-frequency components at the ST-T segment for ischemia analysis. It can be seen from Fig. 10 that if only two KL basis functions are used, the very low frequencies at time 85 ms after the QRS complex will be attenuated. Using three or four basis functions improves the representation of the very low frequencies. The reason for this behavior is the same as explained in Section IV-A; there is a signal interval with higher level of energy (which is more important to the KL transform in order to minimize the mean squared error) than the rest. In this case, the T wave has much more energy than the ST segment.

Another possibility to represent the ST segment with a reduced number of coefficients could be the use of the KL transform applied only to the ST segment (excluding the T wave). In this case, the signal energy is more uniformly distributed, and a better performance should be expected. We show in Fig. 11 the frequency responses at the same time instants of the ST segment, but now, the KL transform is only applied to the ST segment. The low-frequency representation has improved a lot, even in the case of only $p = 1$ KL basis function being used to represent the ST segment.

In summary, values of the number of functions $p = 3$ or 4 should be used in order to represent the low frequencies at the ST segment when the ST-T complex is analyzed with the KL transform. In contrast, fewer functions ($p = 1$) are sufficient to represent the very low-frequency components of the ST segment when only the ST complex is analyzed with the KL transform.

## V. Conclusions

In this correspondence, two different approaches are analyzed for estimating the coefficients of orthogonal expansions using a reduced number of functions: inner product and adaptive estimation with the LMS algorithm. We show that both estimation systems are equivalent to a time-variant periodic filter. Both systems are analyzed by means of their time and frequency descriptions: instantaneous impulse responses $h[m, n]$ and instantaneous frequency responses $H(e^{j\omega}, n)$. The inner product has finite length impulse responses with duration $N$ samples (length of a signal occurrence), whereas impulse responses of the LMS are infinite because of the recursive nature of the LMS algorithm. Both systems have the same frequency response envelopes (if low values of the step-size $\mu$ are used in the LMS), producing a similar lowpass time-variant filtering effect but with the difference that LMS frequency responses have comb shape; therefore, they attenuate uncorrelated noise (nonperiodic with heartbeat occurrence time).

Using the time-variant periodic filter description of orthogonal expansions shown in this work, we can quantitatively know which frequency components are well represented using a reduced number of functions at every time instant of the signal occurrence. Therefore, it can be used as a useful criteria for determining the required number of functions. This description is also a new way to interpret the misadjustment of the LMS algorithm as a frequency response distortion.

Applications can be in data compression, parameter extraction for pattern recognition, detection, and monitoring of pathologies in ECG signal processing, time-variant filter design, etc. Theoretical results are corroborated with the use of the KL transform of the ECG signal, but it can be applied to any orthogonal transform without loss of generality.

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# Multichannel Blind Separation of Sources Algorithm Based on Cross-Cumulant and the Levenberg–Marquardt Method

Ali Mansour and Noboru Ohnishi

***Abstract*—In this correspondence, we derive a new cost function that is more general than our previous one and is based on the cross-cumulant (2 × 2) of the output signals. However, the new algorithm deals with multiple inputs and multiple outputs (MIMO's) and uses a Levenberg–Marquardt method for the minimization of the cost function.**

## I. INTRODUCTION AND CHANNEL MODEL

The blind-separation-of-sources problem involves retrieving the sources from the observations of unknown mixtures of unknown sources. In the general case, authors assume that the sources are non-Gaussian signals and statistically independent of one another. In the general case and in instantaneous mixtures, the fourth-order statistics are required to separate the sources [2], [3].

As shown in Fig. 1, at any time $n$, and with the help of $N$ sensors, $N$ instantaneous mixtures $y_i(n)$ of the $N$ unknown zero-mean sources $x_i(n)$, which are assumed to be statistically independent, are observed. In addition, we assume that the unknown mixture matrix $\boldsymbol{M}$ is a $N \times N$ nonsingular matrix. Taking $Y$ as the mixture vector and $X$ as the source vector (see Fig. 1), we can write $Y = \boldsymbol{M}X$.

The separation is achieved by estimating a $N \times N$ matrix $\boldsymbol{W}$ satisfying $\boldsymbol{WM} = \boldsymbol{\Delta P}$, where $\boldsymbol{P}$ is any permutation matrix, and $\boldsymbol{\Delta}$ is any fullrank diagonal matrix [4]. Let $S$ be the vector of the output signals; then

$$S = \boldsymbol{W}Y = \boldsymbol{G}X \tag{1}$$

where $G = (g_{(i,j)})$ is the global matrix, i.e., $\boldsymbol{G} = \boldsymbol{WM}$. The separation will be performed when $\boldsymbol{G}$ becomes a general permutation matrix (i.e., $\boldsymbol{G} = \boldsymbol{\Delta P}$, where $\boldsymbol{P}$ is a permutation matrix, and $\boldsymbol{\Delta}$ is a fullrank diagonal matrix).

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