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Novel FxLMS convergence condition with deterministic reference

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***Abstract*— A novel analysis of FxLMS convergence when the reference signal is deterministic is presented in this paper. The simple case of a sinusoidal reference is considered first, to be later extended to any combination of multiple sinusoids. In both cases we derive an upper bound for the algorithm step size which ensures convergence. In the derivation of this result there is no need of any of the usual approximations, such as independence between reference and weights or slow convergence, which are not suitable for deterministic references. Instead, we consider the common cases where the adaptive system shows linear time-invariant behavior. The upper bound obtained for the step size is in good agreement with empirical measurements.**

***Index Terms*— Acoustic noise, active noise control, adaptive control, adaptive filters, adaptive signal processing, feedforward systems, least mean square methods, vibration control.**

I. INTRODUCTION

PERIODIC and deterministic noises are very often the subject of cancellation in active noise and vibration control applications. This is due to two reasons: these disturbances are the most annoying and it is usually easier to find a good reference signal to cancel them. However, the adaptive algorithms generally employed in these situations were originally derived considering stochastic signals. This is the case of the filtered reference LMS or FxLMS algorithm [1], [2], which is the most widely used in this context. Therefore, when using this algorithm with deterministic inputs, some behaviors arise that stochastic-based convergence analyses [3], [4] cannot predict. In the case of the LMS algorithm, these behaviors are known as non-Wiener effects [5]–[7].

Moreover, FxLMS convergence analyses with stochastic reference are always based on some assumptions, such as slow convergence or independence between reference signal and filter weights [8]. However, when the reference signal is deterministic, such assumptions are questionable. Specifically, the independence assumption is no longer applicable, whereas the slow convergence assumption compromises the main result we are looking for, that is, a strict upper bound for the adaptation step size to ensure convergence.

In this paper we present a novel convergence analysis for the FxLMS algorithm when the reference signal is deterministic. This analysis is similar to the one made by Glover for the LMS algorithm [5]. It is based on studying the common cases where the adaptive system can be considered to be linear and time-invariant, and applying root locus theory to the system transfer function. Thus, without need of the usual stochastic assumptions, this analysis leads to a reliable bound for the greatest adaptation step size rendering convergence of the FxLMS algorithm with deterministic input. In Section II the simple case of a sinusoidal reference is considered first. Portions of the work introduced in this section were presented in [9]. The results obtained are contrasted with previous analyses and are in good agreement with empirical measurements. The analysis is then extended to simultaneous cancellation of several frequencies: Section III considers the case of multiple sinusoidal references, and Section IV deals with the generic sum of sinusoids as a reference signal. Obviously, this last case comprises any periodic noise as reference, considering Fourier series representation.

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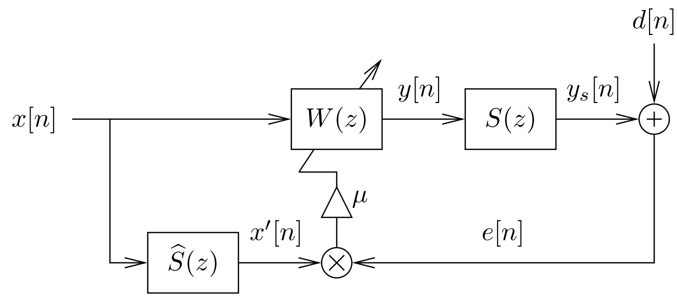

Figure 1: Block diagram of FxLMS algorithm.

II. SINUSOIDAL REFERENCE

The FxLMS algorithm is shown as a block diagram in Fig. 1. In active noise and vibration control, $S(z)$ represents the so-called secondary path, which accounts for the transducer response, the A/D and D/A converters, and the acoustical or structural propagation. $\widehat{S}(z)$ is a model of the secondary path transfer function. The FxLMS algorithm is given by the following set of equations:

$$y[n] = \mathbf{w}^T[n]\mathbf{x}[n], \tag{1a}$$
$$y_s[n] = s[n] * y[n], \tag{1b}$$
$$e[n] = d[n] + y_s[n], \tag{1c}$$
$$\mathbf{x}'[n] = \widehat{s}[n] * \mathbf{x}[n], \tag{1d}$$
$$\mathbf{w}[n+1] = \mathbf{w}[n] - \mu e[n]\mathbf{x}'[n], \tag{1e}$$

where boldface characters represent column vectors.

When the reference signal is sinusoidal, each element of the

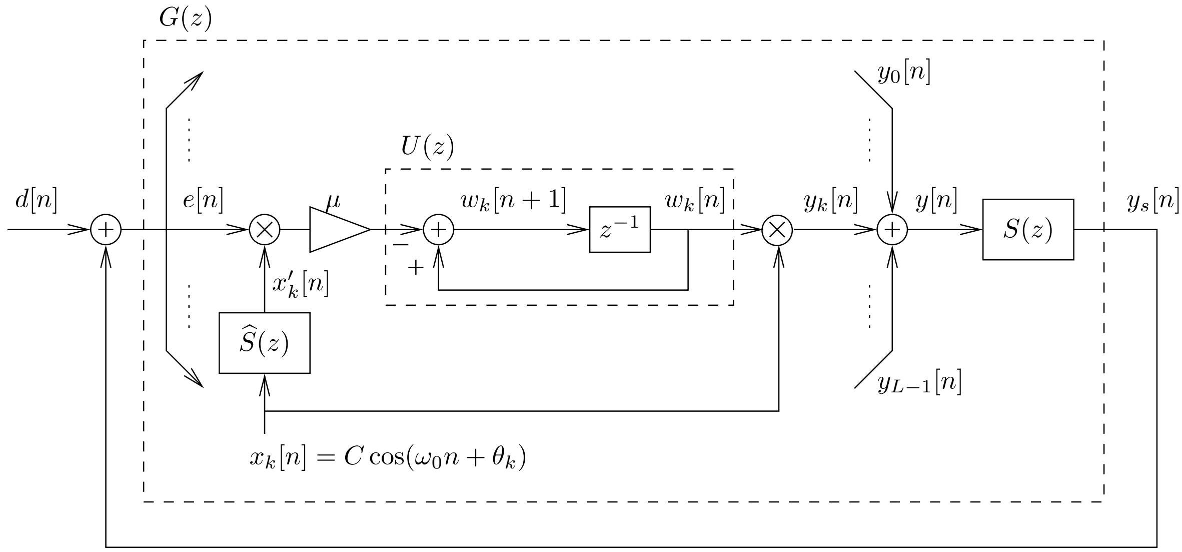

Figure 2: Signal flow diagram for FxLMS algorithm with sinusoidal reference.

reference vector $\mathbf{x}[n]$ admits the following general expression,

$$\begin{aligned} x_k[n] &= C\cos(\omega_0 n + \theta_k) \\ &= \frac{C}{2}\left(e^{j\omega_0 n}e^{j\theta_k} + e^{-j\omega_0 n}e^{-j\theta_k}\right), \end{aligned} \tag{2}$$

where $k = 0, \ldots, L-1$, and L is the total number of filter weights. The FxLMS signal flow diagram, according to (1), is shown in detail for one of these elements $x_k[n]$ in Fig. 2. From this diagram, it is possible to obtain the z-transform of the canceling signal or secondary noise, $Y_s(z)$, as a function of the z-transform of the adaptation error signal or residual noise, $E(z)$. Thus, we can find the open-loop input-output transformation for the feedback system shown in Fig. 2.

This result has already been obtained by Elliott and Nelson [10], [11] for the case of a synchronously sampled sinusoid, that is, $\omega_0 = i\pi/L$, with integer i. However, we include here the detailed derivation of the single-sinusoid input-output transformation in order to facilitate the more complex derivations of the subsequent multiple-sinusoid cases.

A. Open-loop input-output transformation

Considering (2) and the exponential multiplication property of the z-transform, the kth weight of the adaptive filter can be expressed in the transform domain as

$$\begin{aligned} W_k(z) = \frac{C}{2}\mu\Big[&\widehat{S}(e^{j\omega_0})e^{j\theta_k}E(ze^{-j\omega_0}) \\ &+ \widehat{S}(e^{-j\omega_0})e^{-j\theta_k}E(ze^{j\omega_0})\Big]U(z), \end{aligned} \tag{3}$$

where

$$U(z) = \frac{-1}{z-1} \tag{4}$$

is the transfer function of the inner dashed block in Fig. 2. The contribution of this kth weight to the adaptive filter output is given by

$$\begin{aligned} Y_k(z) &= \frac{C}{2}\left[e^{j\theta_k}W_k(ze^{-j\omega_0}) + e^{-j\theta_k}W_k(ze^{j\omega_0})\right] \\ &= \frac{C^2}{4}\mu\Big[\widehat{S}(e^{-j\omega_0})U(ze^{-j\omega_0})E(z) \\ &\quad + \widehat{S}(e^{j\omega_0})U(ze^{j\omega_0})E(z) \\ &\quad + \widehat{S}(e^{j\omega_0})e^{j2\theta_k}U(ze^{-j\omega_0})E(ze^{-j2\omega_0}) \\ &\quad + \widehat{S}(e^{-j\omega_0})e^{-j2\theta_k}U(ze^{j\omega_0})E(ze^{j2\omega_0})\Big]. \end{aligned} \tag{5}$$

Combining all of the $Y_k(z)$ components we get the control signal output of the adaptive filter, $Y(z)$. Eventually, the secondary noise signal is obtained as filtering $Y(z)$ by the secondary path,

$$Y_s(z) = S(z)Y(z) = S(z)\sum_{k=0}^{L-1} Y_k(z). \tag{6}$$

Substituting for $Y_k(z)$ and rearranging yields the open-loop input-output transformation we were searching for,

$$\begin{aligned} Y_s(z) = &\frac{C^2}{4}\mu L\Big[\widehat{S}(e^{-j\omega_0})U(ze^{-j\omega_0}) \\ &\quad + \widehat{S}(e^{j\omega_0})U(ze^{j\omega_0})\Big]S(z)E(z) \\ &+ \frac{C^2}{4}\mu\Bigg[\widehat{S}(e^{j\omega_0})U(ze^{-j\omega_0})E(ze^{-j2\omega_0})\sum_{k=0}^{L-1}e^{j2\theta_k} \\ &+ \widehat{S}(e^{-j\omega_0})U(ze^{j\omega_0})E(ze^{j2\omega_0})\sum_{k=0}^{L-1}e^{-j2\theta_k}\Bigg]S(z). \end{aligned} \tag{7}$$

The first two terms in (7) represent the time-invariant part of the response from $E(z)$ to $Y_s(z)$, since they fulfill the convolution theorem, and, so, only frequencies of $E(z)$ appear at the output. On the contrary, the last two terms in (7) are time-varying, since they introduce unwanted frequency shifted components of $E(z)$ at the output $Y_s(z)$.

Next, we comment on two special cases of sinusoidal reference, which are also very common and with great relevance.

1) In-phase and quadrature (I/Q) sinusoidal components. In this case, there are only two sinusoidal components in the reference vector, with a phase shift between them of $\pi/2$ rad:

$$\mathbf{x}[n] = \left(\begin{array}{c} x_0[n] \\ x_1[n] \end{array} \right) = \left(\begin{array}{c} C\cos(\omega_0 n + \theta) \\ C\sin(\omega_0 n + \theta) \end{array} \right). \quad (8)$$

So, the filter length is $L = 2$, with $\theta_0 = \theta$ and $\theta_1 = \theta - \pi/2$, yielding $\sum_{k=0}^{L-1} e^{\pm j2\theta_k} = 0$. Thus, in this case, the time-varying terms in (7) are exactly zero.

2) Transversal filter. When a tapped-delay line is used with a sinusoidal reference input, the initial phase of each component of the reference vector is given by $\theta_k = \theta - \omega_0 k$, and so

$$\sum_{k=0}^{L-1} e^{\pm j2\theta_k} = e^{\pm j[2\theta - \omega_0(L-1)]} \frac{\sin(\omega_0 L)}{\sin(\omega_0)}. \quad (9)$$

In this case, when the frequency of the sinusoid is $\omega_0 = i\pi/L$, with integer i, (9) is exactly zero and, consequently, so again are the time-varying terms in (7). In addition, for any frequency ω_0, when the number of filter weights of the transversal filter L is sufficiently high, (9) approaches zero, and the time-varying terms in (7) may be considered negligible, even though not being exactly zero.

In the previous cases, and in any other case where the time-varying terms in (7) are zero or negligible, the response from $E(z)$ to $Y_s(z)$ is linear and time-invariant (LTI). Therefore, we can define the following *open-loop transfer function*,

$$G(z) = \frac{Y_s(z)}{E(z)} = -\frac{C^2}{4}\mu L \left[\frac{\widehat{S}(e^{-j\omega_0})}{ze^{-j\omega_0} - 1} + \frac{\widehat{S}(e^{j\omega_0})}{ze^{j\omega_0} - 1} \right] S(z). \quad (10)$$

The secondary path model is regularly a real system, and thus its frequency response is conjugate-symmetric, $\widehat{S}(e^{j\omega}) = \widehat{S}^*(e^{-j\omega})$. Taking this into account, the open-loop transfer function can be expressed [10] [1, p. 126] as follows ,

$$G(z) = -\frac{C^2}{2}\mu L \left| \widehat{S}_{\omega_0} \right| \left[\frac{\cos(\omega_0 - \phi_{\omega_0})z - \cos(\phi_{\omega_0})}{z^2 - 2\cos(\omega_0)z + 1} \right] S(z), \quad (11)$$

where $|\widehat{S}_{\omega_0}| = |\widehat{S}(e^{j\omega_0})|$ and $\phi_{\omega_0} = \angle\widehat{S}(e^{j\omega_0})$.

B. Root locus analysis of the closed-loop transfer function

The *closed-loop transfer function*, from the primary noise $D(z)$ to the residual noise $E(z)$, is easily obtained from $G(z)$,

$$H(z) = \frac{E(z)}{D(z)} = \frac{1}{1 - G(z)}. \quad (12)$$

Therefore, in the special but very common cases where the adaptive system exhibits LTI behavior, the upper bound for the step size to ensure convergence can be obtained analyzing the stability of this transfer function, $H(z)$, without needing any questionable assumption.

The analysis of the most general case, with any secondary path $S(z)$ and any model $\widehat{S}(z)$, is so difficult that it is almost impossible to extract any global conclusion. Therefore, in the following, we consider the simple case where the secondary path is composed of a pure delay[1] and a gain factor, that is, $S(z) = Az^{-\Delta}$. We also consider perfect modeling of this secondary path, $\widehat{S}(z) = S(z)$. In this case, the open-loop transfer function, from (11), is given by

$$G(z) = -P_{x'} L\mu \frac{\cos[\omega_0(\Delta+1)]z - \cos(\omega_0\Delta)}{z^{\Delta}[z^2 - 2\cos(\omega_0)z + 1]}, \quad (13)$$

where $P_{x'} = C^2A^2/2$ is the power of the filtered reference signal, $x'[n]$. This function can be expressed as

$$G(z) = -\tilde{\mu}\frac{N(z)}{D(z)} \quad (14)$$

where the gain factor

$$\tilde{\mu} = P_{x'} L\mu \quad (15)$$

is the normalized step size, and $N(z) = \cos[\omega_0(\Delta+1)]\, z - \cos(\omega_0\Delta)$ and $D(z) = z^{\Delta}\left[z^2 - 2\cos(\omega_0)z + 1\right]$ are polynomials in z. So, from (12), the closed-loop transfer function is

$$H(z) = \frac{D(z)}{D(z) + \tilde{\mu}N(z)}. \quad (16)$$

From (16), it is clear that the poles of $G(z)$ are simultaneously zeros of $H(z)$. So, the adaptive system with sinusoidal reference behaves as a notch filter at the frequency of the reference, since $e^{\pm j\omega_0}$ are the poles of $G(z)$ and so, zeros of $H(z)$. On the other hand, the poles of $H(z)$ are the $\Delta + 2$ roots of the characteristic equation

$$D(z_p) + \tilde{\mu}N(z_p) = 0. \quad (17)$$

As long as the modulus of all of these roots is less than unity, $|z_p| < 1$, the adaptive system will be stable, that is to say, will converge. Thus, root locus analysis [12] of the characteristic equation (17) makes it possible to obtain the values for the normalized step size $\tilde{\mu} = P_{x'}L\mu$ that ensures stability of the system.

The following conclusions are extracted from this analysis:

- When $\tilde{\mu} = 0$, two of the roots from (17) are $z_p = e^{\pm j\omega_0}$, that is, they are on the unit circle, and all of the others are $z_p = 0$. In this trivial case, without adaptation, $H(z) = 1$.
- With negative $\tilde{\mu}$, at least the two roots that were lying on the unit circle go outside, turning the system unstable, as could be expected.
- With positive and sufficiently small $\tilde{\mu}$, all of the $\Delta + 2$ roots are inside the unit circle, and so, the system is stable. An example of root loci is shown in Fig. 3 for $\omega_0 = \pi/4$ and $\Delta = 5$, when the normalized step size $\tilde{\mu}$ varies from 0 to 1. The arrows indicate the direction of increasing values for $\tilde{\mu}$.
- There is an upper bound for the normalized step size $\tilde{\mu}$, depending on both the frequency of the reference, ω_0, and the secondary path delay, Δ. When $\tilde{\mu} > \tilde{\mu}_{\max}(\omega_0, \Delta)$, there is at least one root outside the unit circle, which again turns the system unstable. In the example shown in Fig. 3, $\tilde{\mu}_{\max}(\pi/4, 5) = 0.45$. For this reason, some of the

[1]When the secondary path is just a pure delay, the FxLMS algorithm is equivalent to the simpler delayed LMS or DLMS.

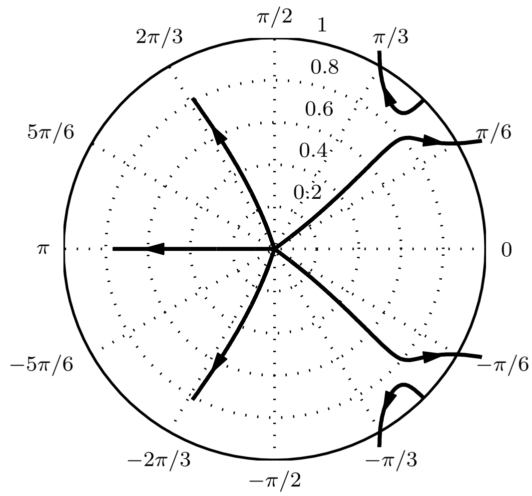

Figure 3: Root loci for $\omega_0 = \pi/4$ and $\Delta = 5$, for $0 \le \tilde{\mu} \le 1$.

branches in the root loci go across the unit circle, since the normalized step size $\tilde{\mu}$ varies from 0 to 1.

Therefore, the convergence condition for the adaptive system is always $0 < \tilde{\mu} < \tilde{\mu}_{\max}(\omega_0, \Delta)$. Fig. 4 displays the stability upper bound for the normalized step size $\tilde{\mu}$ as a function of frequency for some particular values of the secondary path delay.

Even though there seems to be a clear pattern in the curves of $\tilde{\mu}_{\max}(\omega_0, \Delta)$, it is not simple at all to obtain a closed-form analytical expression. In any case, the frequency of the reference may be unknown before turning on the adaptive system or could be varying. For this reason, it seems useful to obtain an upper bound for the normalized step size to ensure convergence for every possible frequency. It can be seen in Fig. 4 that for a given delay Δ in the secondary path, the minimum value of the upper bound $\tilde{\mu}_{\max}$, ensuring stability for every frequency ω_0 in the reference, is reached when $\omega_0 \to 0$ or $\omega_0 \to \pi$. In the first case, when $\omega_0 \to 0$, system stability is lost because one of the poles of $H(z)$ goes across the unit circle through $z_p = 1$. When $\omega_0 \to \pi$, the crossing point is $z_p = -1$. The upper bound for $\tilde{\mu}$ may be obtained from (17) considering that $\omega_0 \to 0$ and $z_p = 1$, or alternatively, when $\omega_0 \to \pi$ and $z_p = -1$. Thus, we get

$$\begin{aligned} \tilde{\mu}_{\max}(\Delta) &= \lim_{\omega_0 \to 0} -\frac{D(1)}{N(1)} \\ &= \lim_{\omega_0 \to \pi} -\frac{D(-1)}{N(-1)} \\ &= \frac{2}{2\Delta + 1}. \end{aligned} \tag{18}$$

Since $\tilde{\mu} = P_{x'} L \mu$, the convergence condition for the step size, without normalization, as a function of secondary path delay, but ensuring convergence for every frequency, is eventually given by

$$0 < \mu < \frac{2}{P_{x'} L (2\Delta + 1)}. \tag{19}$$

C. Comparison with previous analyses

In the case of a white reference signal, the valid range usually considered for the step size is [2], [13]

$$0 < \mu < \frac{2}{P_{x'} (L + \Delta)}. \tag{20}$$

Comparing (19) and (20), we note that the convergence condition in the sinusoidal reference case is much more restrictive than in the white reference case. With a sinusoidal reference, the upper bound for the step size is inversely proportional to the product of the length of the filter and the delay in the secondary path, whereas with a white reference signal we get only the sum of these parameters, instead of their product.

In [3], Bjarnason analyzes FxLMS convergence with a sinusoidal reference, but employs the habitual assumptions made with stochastic signals, that is, independence theory. The stability condition derived in that analysis is as follows,

$$0 < \mu < \frac{2}{P_{x'} L} \sin\left(\frac{\pi}{2(2\Delta + 1)}\right). \tag{21}$$

In the event of large delay Δ in the secondary path, (21) simplifies to

$$0 < \mu < \frac{\pi}{P_{x'} L (2\Delta + 1)}. \tag{22}$$

The similarity between this last convergence condition and the one we have just derived in (19) is evident. Nevertheless, it has to be pointed out that our analysis is exact, at least for all the cases where the time-varying terms of the open-loop response in (7) are negligible compared to the time-invariant terms.

It is also interesting to note that the stability range (19) is also valid for the LMS algorithm, since it can be seen as a particular case of the FxLMS algorithm with $\Delta = 0$. Thus, the upper bound for the LMS from (19) is exactly the same as already obtained by Glover [5].

Some authors have considered DLMS convergence with a sinusoidal reference, but only for the particular case where $\omega_0 = \pi/2$. According to Elliott, Stothers, and Nelson [11, eq. (29)], the optimum step size for a filter with two coefficients and $P_x = 1/2$ is $\mu_{\mathrm{o}} \approx 1/(1.35\Delta)$. For these authors, the optimum step size is the greatest value for μ without oscillatory behavior in the learning curve, which will obviously be lower than the stability upper bound. Also, Morgan and Sandford [14] establish a stability upper bound for the step size, $\mu \lesssim \pi/\Delta$, for the same situation, $L = 2$ and $P_x = 1/2$.

In order to facilitate comparison with these results, we consider next in our analysis the stability when the reference frequency is $\omega_0 = \pi/2$. Again applying root locus theory to the characteristic equation (17), it can be shown that the maximum value for the normalized step size $\tilde{\mu}$ yielding a stable adaptive system is

$$\tilde{\mu}_{\max}\big|_{\omega_0 = \frac{\pi}{2}} = 2 \sin\left[\frac{\pi}{2\left(2\left\lfloor \Delta/2 \right\rfloor + 1\right)}\right], \tag{23}$$

where $\lfloor a \rfloor$ (“floor”) stands for the rounding function returning the greatest integer less than or equal to a. For large delay Δ, we can approximate this as

$$\tilde{\mu}_{\max}\big|_{\omega_0 = \frac{\pi}{2}} \approx \frac{\pi}{\Delta}. \tag{24}$$

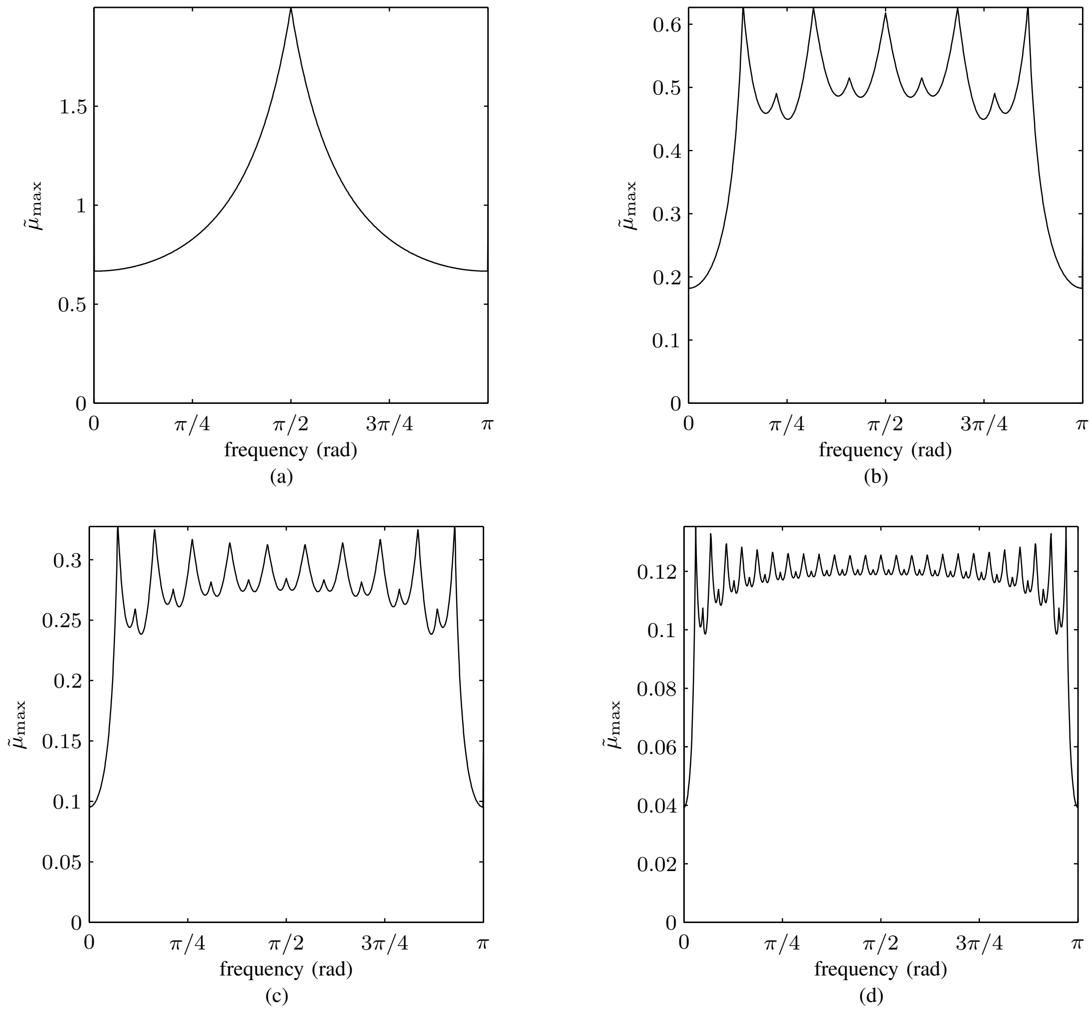

Figure 4: Upper bound for the normalized step size $\tilde{\mu} = P_{x'} L \mu$ as a function of the reference frequency ω_0, for several values of secondary path delay. (a) $\Delta = 1$, (b) $\Delta = 5$, (c) $\Delta = 10$, (d) $\Delta = 25$.

Therefore, when $\omega_0 = \pi/2$, the stability bound for the step size, without normalization, is

$$\mu_{\max}\big|_{\omega_0 = \frac{\pi}{2}} = \frac{2}{P_{x'} L} \sin\left[\frac{\pi}{2\left(2\lfloor\Delta/2\rfloor + 1\right)}\right] \approx \frac{\pi}{P_{x'} L \Delta}. \tag{25}$$

Considering the particular case where $L = 2$ and $P_{x'} = 1/2$, we see that the upper bound in (25) is in close agreement with the ones already commented on from previous analyses [11], [14] for sinusoidal references with frequency $\omega_0 = \pi/2$.

The convergence condition (19) could be seen as rather conservative, due to the fact of being valid for every frequency. In fact, inspecting the condition for the particular case of $\omega_0 = \pi/2$, which is the mid-point in the curves in Fig. 4, there is an approximate factor of π between both convergence conditions, (19) and (25). However, we also see that in (25) there is still a relation of inverse proportionality with the product of the length of the filter L and the delay introduced by the secondary path Δ.

In our analysis we have only considered the case of noiseless sinusoidal references. Some authors have analyzed the LMS algorithm with noisy sinusoidal reference [15], [16]. The main conclusion from these analyses is that the adaptive system will no longer behave as a linear time-invariant system due to the presence of noise in the reference. However, for reasonable signal-to-noise ratios it seems that the effect of this noise is insignificant.

D. Empirical validation

In order to check the validity of the upper bound for the step size found in our analysis, several experiments have been carried out. Fig. 5 shows some empirical results together with the theoretical prediction obtained with root locus theory and the LTI approximation. These results correspond to the empirical upper bounds for the normalized step size when transversal filters with $L = 20$ and $L = 2$ coefficients are used. The different frequencies considered for the sinusoidal reference are $\omega = i\pi/50$, with integer i ranging from 1 to 49.

When we consider the transversal filter with $L = 20$ coefficients, we can see that there is good agreement between theoretical prediction and the empirical results. Of course, the theoretical prediction is not exact, since it is based on the LTI approximation. In fact, the approximation is exact only for frequencies $\omega = i\pi/50$ with i being an integer multiple of 5. For these frequencies, we check that the empirical bound

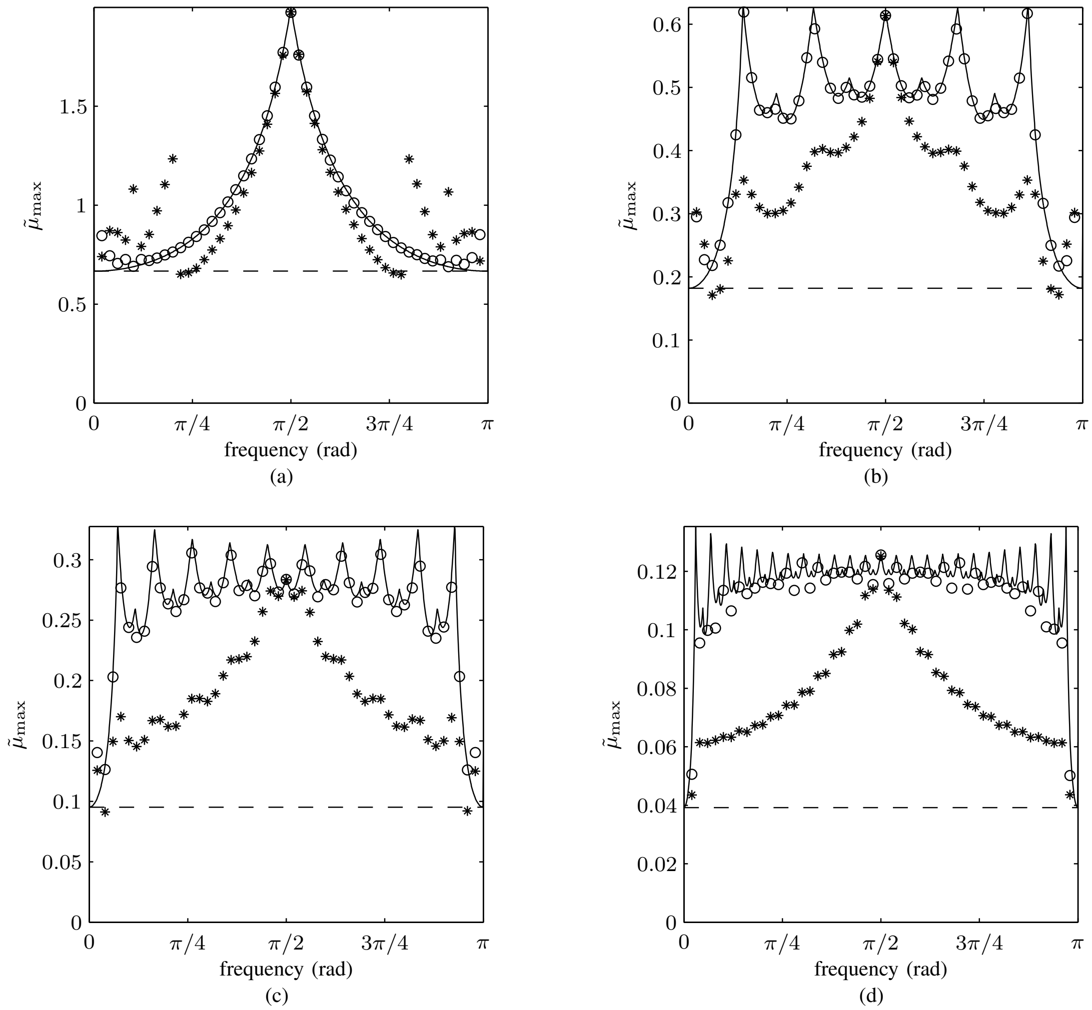

Figure 5: Upper bound for the normalized step size $\tilde{\mu} = P_{x'} L \mu$ as a function of the reference frequency ω_0, for several values of secondary path delay: theoretical prediction (solid), empirical results with $L = 20$ (circles) and $L = 2$ (asterisks), and overall-frequency bound (dashed). (a) $\Delta = 1$, (b) $\Delta = 5$, (c) $\Delta = 10$, (d) $\Delta = 25$.

really lies on the theoretical curve. However, for the rest of the frequencies, there is little difference between the theoretical prediction and the empirical bound.

For the sake of comparison, we consider the case of Fig. 5(d), with $\Delta = 25$ and $L = 20$. The upper bound we have derived, that is, the minimum value of the theoretical curve, is in this case $\mu_{\max} = 2/(P_{x'} 1020)$. If we make use of the usual bound (20) derived for a white reference signal, the upper bound would be $\mu_{\max} = 2/(P_{x'} 45)$, that is, more than 22 times greater. Hence, the bound in (19) seems much more appropriate, even though it may be considered a bit conservative, as we have already commented.

When the transversal filter has only $L = 2$ coefficients, the agreement between the empirical bounds and the theoretical prediction is not so good. However, observe that this is the worst case from the point of view of the LTI approximation, since there is only one frequency, $\omega_0 = \pi/2$, for which we can say that the adaptive system behaves as being LTI. At all of the other frequencies, the time-varying terms in (7) are not zero. This is the only reason for the differences found between theoretical prediction and the empirical results. In fact, if we consider the case of $L = 2$ filter weights but with I/Q sinusoidal components (not shown in the graphics), where the LTI approximation is valid for every frequency, the match between predicted and empirical bounds is perfect.

Nevertheless, despite the differences caused by the applicability of the LTI approximation, as shown in Fig. 5, the convergence condition (19) seems a really good one, even in this worst case: the value of the minimum empirical upper bound is, for every secondary delay, very close to the theoretical one, although these minima do not really occur at the same frequency.

III. Multiple sinusoidal references

In this section, we consider the case of multiple reference signals that are independently processed. That is to say, there is an adaptive filter for each reference signal, and the outputs of all of the filters are summed to form the control signal $y[n]$. Each of these reference signals is a sinusoid of frequency ω_i. We can think of using an in-phase and quadrature component for each sinusoid or, alternatively, a transversal filter, with a number of coefficients L_i sufficiently high, to process each sinusoid. For both situations, the behavior of the open-loop system for each frequency is assumed to be linear and time-invariant, as discussed in the previous section. Therefore, we can define the ith transfer function

$$\begin{aligned}G_i(z) &= -P_{x_i'} L_i \mu_i \frac{\cos[\omega_i(\Delta+1)]z - \cos(\omega_i \Delta)}{z^{\Delta}[z^2 - 2\cos(\omega_i)z + 1]} \\ &= -\tilde{\mu}_i \frac{N_i(z)}{D_i(z)},\end{aligned} \tag{26}$$

where

$$\tilde{\mu}_i = P_{x_i'} L_i \mu_i, \tag{27}$$

and $N_i(z)$ and $D_i(z)$ are the polynomials in z from the numerator and denominator, respectively. Due to the presence of multiple reference signals, the global open-loop transfer function is now the sum of all of these individual contributions,

$$G(z) = \sum_{i=1}^{N_{\mathrm{ref}}} G_i(z) = -\sum_{i=1}^{N_{\mathrm{ref}}} \tilde{\mu}_i \frac{N_i(z)}{D_i(z)}, \tag{28}$$

where N_{ref} is the number of independent reference signals. Using the relation (12) yields, also in this case, the closed-loop transfer function,

$$H(z) = \frac{1}{1 + \sum_{i=1}^{N_{\mathrm{ref}}} \tilde{\mu}_i \frac{N_i(z)}{D_i(z)}}. \tag{29}$$

Analyzing the stability of $H(z)$, we can get an upper bound for the algorithm step size. It should be pointed out that in this case, we could use different step sizes, μ_i, for each of the multiple-reference signals. However, it seems sensible that for every reference signal, the maximum value for the normalized step size $\tilde{\mu}_i$ is the same. Thus, taking $\tilde{\mu}_{i,\max} = \tilde{\mu}_{\max}$ makes the analysis much simpler.

The maximum normalized step size $\tilde{\mu}_{\max}(\Delta)$, to ensure convergence for every possible set of sinusoidal references, will be the real and positive minimum value of $-1/\sum_{i=1}^{N_{\mathrm{ref}}} N_i(e^{j\omega})/D_i(e^{j\omega})$. For each of the terms $-N_i(e^{j\omega})/D_i(e^{j\omega})$, the maximum positive and real value is obtained when $\omega = 0$ and $\omega_i \to 0$, or alternatively, when $\omega = \pi$ and $\omega_i \to \pi$. Therefore, the worst case for the stability of $H(z)$ occurs also when one of the poles crosses the unit circle through $z = 1$ when $\omega_i \to 0$, or crosses through $z = -1$ when $\omega_i \to \pi$. Thus, we eventually find the upper bound,

$$\begin{aligned}\tilde{\mu}_{\max}(\Delta) &= \lim_{\omega_i \to 0} -\frac{1}{\sum_{i=1}^{N_{\mathrm{ref}}} \frac{N_i(1)}{D_i(1)}} \\ &= \frac{2}{N_{\mathrm{ref}}(2\Delta+1)}.\end{aligned} \tag{30}$$

Therefore, stability is guaranteed for each of the reference signals when

$$0 < \mu_i < \frac{2}{N_{\mathrm{ref}} P_{x_i'} L_i (2\Delta+1)}. \tag{31}$$

Comparing this last result with the convergence condition obtained for a single sinusoidal reference, (19), we note that the maximum step size has been reduced by N_{ref}, and the only reason for this is having simultaneously several sinusoidal signals as references.

IV. Single multi-frequency reference

Our initial analysis for one sinusoidal reference can also be easily extended to the case of a reference signal consisting of the sum of several sinusoids [5]. Let $N_{\sin}$ be the total number of sinusoids in the reference,

$$x[n] = \sum_{i=1}^{N_{\sin}} C_i \cos(\omega_i n + \theta_i). \tag{32}$$

Now we consider only the case of a transversal filter. So, the kth component of the reference signal vector is

$$x_k[n] = \sum_{i=1}^{N_{\sin}} C_i \cos[\omega_i(n-k) + \theta_i]. \tag{33}$$

Proceeding in the same way as before for a single sinusoid, we get the following expression for the secondary noise,

$$\begin{aligned}Y_s(z) = &-\mu L \sum_{i=1}^{N_{\sin}} \frac{C_i^2}{2} \left|\widehat{S}(e^{j\omega_i})\right| Q_i(z) S(z) E(z) \\ &+\mu \left\{ \sum_{i=1}^{N_{\sin}} \sum_{\substack{j=1 \\ j\neq i}}^{N_{\sin}} \frac{C_i C_j}{4} \frac{\sin\left(\frac{\omega_i - \omega_j}{2} L\right)}{\sin\left(\frac{\omega_i - \omega_j}{2}\right)} [\mathrm{TV}] \right. \\ &\left. + \sum_{i=1}^{N_{\sin}} \sum_{j=1}^{N_{\sin}} \frac{C_i C_j}{4} \frac{\sin\left(\frac{\omega_i + \omega_j}{2} L\right)}{\sin\left(\frac{\omega_i + \omega_j}{2}\right)} [\mathrm{TV}] \right\} S(z),\end{aligned} \tag{34}$$

where

$$Q_i(z) = \frac{\cos(\omega_i - \phi_{\omega_i})z - \cos(\phi_{\omega_i})}{z^2 - 2\cos(\omega_i)z + 1}, \tag{35}$$

and $\phi_{\omega_i} = \angle \widehat{S}(e^{j\omega_i})$. In (34), TV represents time-varying frequency-shifted components of the error signal $E(z)$. Therefore, the first term in (34) is the time-invariant part of the open-loop response and the last terms are the time-varying part. For these last terms to be negligible when compared to the time-invariant response, we must have

$$\frac{\sin\left(\frac{\omega_i \pm \omega_j}{2} L\right)}{\sin\left(\frac{\omega_i \pm \omega_j}{2}\right)} \ll L. \tag{36}$$

Consequently, the filter length required for achieving LTI behavior from the adaptive system may be in this case quite high. Specifically, large L will be required when some of the frequencies of the different sinusoids are very close.

When we have LTI behavior and consider the simple secondary path $S(z) = Az^{-\Delta}$, the open-loop transfer function is

$$G(z) = -\mu L A^2 \sum_{i=1}^{N_{\sin}} \frac{C_i^2}{2} \frac{\cos[\omega_i(\Delta+1)]z - \cos(\omega_i\Delta)}{z^{\Delta}[z^2 - 2\cos(\omega_i)z + 1]}. \tag{37}$$

So, in this case we find the same open-loop transfer function as that of the multiple sinusoidal references (28). Nevertheless, in this situation the step size μ and the filter length L are unique, since there is just one reference signal.

The worst case from the viewpoint of stability is again a pole of the closed-loop transfer function going out of the unit circle through $z = 1$ when $\omega_i \to 0$. Thus, convergence of the adaptive system is now guaranteed as long as

$$0 < \mu < \frac{2}{L(2\Delta + 1)A^2 \sum_{i=1}^{N_{\sin}} C_i^2/2} = \frac{2}{P_{x'}L(2\Delta + 1)}. \quad (38)$$

Comparing (38) with (19), we see that when the reference signal is a generic sum of sinusoids, the sinusoidal stability upper bound is still valid.

V. Conclusions

The FxLMS convergence analysis presented in this paper has obtained a strict upper bound on the algorithm step size when the reference signal is deterministic. Several cases have been considered in detail: single sinusoidal reference, multiple sinusoidal references, and single multi-frequency reference. The analysis is founded on considering the cases where, with a deterministic reference, the adaptive system global behavior is linear and time-invariant. Applying root locus theory to the transfer function of the LTI adaptive system, the maximum value of the algorithm step size for which the system is stable is determined. Thus, the usual assumptions of stochastic convergence analyses have been avoided, such as independence between filter weights and reference signal or slow convergence.

The upper bound obtained for deterministic references is clearly much more restrictive than the one generally considered for stochastic wideband references. With a white reference, the maximum stable step size is inversely proportional to the *sum* of the length of the filter and the delay in the secondary path. However, when the reference is deterministic, the upper bound is inversely proportional to the *product* of these two parameters. Hence, this new upper bound is more accurate and should be the one considered whenever the reference is deterministic, since the stochastic reference bound would easily lead to divergence.

The convergence condition derived for a deterministic reference is also in good agreement with special cases of previous analyses. Furthermore, empirical observations clearly support the theoretical results, even though the LTI approximation is not always strictly applicable.

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