AN IRWLS PROCEDURE FOR ROBUST BEAMFORMING WITH SIDELOBE CONTROL

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ABSTRACT

Recently, the application of support vector machines (SVM) has been proposed for increasing the robustness of array beamforming against signal mismatch situations and providing additional sidelobe control. In this approach, the conventional Capon cost function is modified by including a regularization term that penalizes differences between the actual and the ideal array responses. The resulting cost function is convex with a unique global minimum that can be found using quadratic programming (QP) techniques. This paper expands this approach in order to reduce the computational cost of the SVM solution by means of an iterative re-weighted least squares (IRWLS) procedure. The robustness of the proposed beamformer is examined in several mismatch scenarios, showing a good performance even when the signal DOA mismatch error is underestimated: a situation where other robust approaches typically fail.

1. INTRODUCTION

Increasing the robustness of adaptive beamforming techniques has focused considerable efforts during the recent years. Several works have shown that the traditional minimum variance distortionless response (MVDR) or Capon beamformer degrades considerably its performance in practical scenarios with limited number of available snapshots, mismatches between the assumed direction of arrival and the actual one, or calibration errors.

Several approaches have been proposed to improve the robustness of the MVDR beamformer, for instance, diagonal loading methods [1][2], and eigenspace-based beamformers [3]. In addition, when the number of snapshots used for covariance matrix estimation is insufficient or the scenario is favourable in terms of SNR, the MVDR beamformer can present unacceptably high sidelobes, which reduces its performance in the presence of unexpected interferences. To overcome this drawback, sidelobe control has been proposed in [4] and [5]. These approaches modify the MVDR beamforming problem to incorporate additional constraints outside the mainlobe beampattern. In [4], the corresponding optimization problem is solved by second-order cone (SOC) programming techniques. If a feasible solution does not exist, the sidelobe levels defining the constraints must be relaxed and a new optimization problem must be solved. Moreover, this technique does not consider the signal mismatch problem.

In [5], the design of a robust adaptive beamformer is approached from the support vector machine (SVM) framework [6]. The MVDR beamforming problem is reformulated by incorporating an additional regularization term that penalizes sidelobe levels while, at the same time, allows a certain error in the desired signal direction. The resulting cost function can be interpreted as a support vector machine for regression (SVR), whose unique solution has traditionally been found using quadratic programming techniques. In comparison to [4], the proposed robust SVM-based beamformer always provides a solution even if the problem is not feasible.

In this paper the SVM beamformer is obtained by means of an iterative re-weighted least square (IRWLS) algorithm, which considerably reduces the complexity of the conventional quadratic programming techniques. This procedure has been successfully applied to solve SVM's [7] and it has recently proven to converge to the SVM solution [8]. We present simulation examples where the performance of the proposed SVM beamformer solved via IRWLS is compared with the Capon beamformer and with other robust beamforming techniques. The IRWLS beamformer exhibits an increased robustness against the signal mismatch problem, even when the DOA estimation error is larger than expected.

2. BACKGROUND

Consider a narrowband beamformer with M sensors, its output is given by

$$y(k) = \mathbf{w}^H \mathbf{x}(k),$$

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where k is the time index, $\mathbf{x}(k) = [x_1(k), \ldots, x_M(k)]^T \in C^{M\times 1}$ is the complex vector of array observations, $\mathbf{w} = [w_1, \ldots, w_M]^T \in C^{M\times 1}$ is the complex vector of beamformer weights, and $(.)^T$ and $(.)^H$ denote the transpose and Hermitian transpose, respectively. The observation (snapshot) vector is given by

$$\mathbf{x}(k) = s(k)\mathbf{a}(\theta_s) + \sum_{j=1}^{N_i} i_j(k)\mathbf{a}(\theta_j) + \mathbf{n}(k),$$

where $s(k)$ and $i_j(k)$ are the signal of interest (SOI) and interference waveforms; $\mathbf{n}(k)$ represents the noise signal. The signal and interference directions of arrival (DOA) are θ_s and θ_j, $j = 1, \ldots, N_i$, respectively, with corresponding steering vectors $\mathbf{a}(\theta_s)$ and $\mathbf{a}(\theta_j)$.

The classical formulation for the MVDR beamformer is

$$\min_{\mathbf{w}} E\left[|y(k)|^2\right] \qquad \text{subject to} \quad \mathbf{w}^H\mathbf{a}(\theta_s) = g \tag{1}$$

whose solution is given by

$$\mathbf{w}_0 = \frac{g^*\mathbf{R}_x^{-1}\mathbf{a}(\theta_s)}{\mathbf{a}(\theta_s)^H\mathbf{R}_x^{-1}\mathbf{a}(\theta_s)}, \tag{2}$$

where * denotes conjugate. In practice, the exact covariance $M \times M$ matrix $\mathbf{R}_x$ is not available and is replaced by the sampled covariance matrix $\hat{\mathbf{R}}_x$ which is given by

$$\hat{\mathbf{R}}_x = \frac{1}{N}\sum_{k=1}^{N}\mathbf{x}(k)\mathbf{x}(k)^H, \tag{3}$$

where N is the number of observed snapshots.

3. ROBUST BEAMFORMING WITH SIDELOBE CONTROL

In this section, the conventional MVDR beamforming problem is modified by incorporating additional constraints in order to increase the robustness against mismatches in the SOI steering vector, as well as to control the sidelobe level. Likewise [4], the additional constraints are included as a regularization term of the array output power [5].

Let us consider a grid of directions of arrival θ_i, $i = 1, \cdots, P$; which sample the beampattern in $[-90^o, 90^o]$. We define an angular mainlobe beamwidth $\triangle$ centered at the assumed SOI direction of arrival θ_s. P_1 from the total set of angles sample the mainlobe beamwidth, including θ_s: in this way, we account for a possible signal mismatch error. The remaining $P_2 = P - P_1$ angles sample the beampattern outside the mainlobe. Based on the formulation of SVMs [6], we consider the following regularized MVDR problem

$$\min_{\mathbf{w}} \quad \frac{1}{2}\mathbf{w}^H\mathbf{R}_x\mathbf{w} + C\sum_{i=1}^{P}|d_i - \mathbf{w}^H\mathbf{a}(\theta_i)|_\epsilon, \tag{4}$$

where

$$|d_i - \mathbf{w}^H\mathbf{a}(\theta_i)|_\epsilon = \max\left\{0, |d_i - \mathbf{w}^H\mathbf{a}(\theta_i)| - \epsilon\right\}, \tag{5}$$

is the so-called Vapnik's ϵ-insensitive loss function [6] and d_i is the desired beamformer output

$$d_i = \begin{cases} 0 & \text{if } |\theta_i - \theta_s| > \triangle, \\ g_R + jg_I & \text{if } |\theta_i - \theta_s| \le \triangle, \end{cases} \tag{6}$$

Note that the regularized cost function (4) establishes a trade-off between the array output power and a term that penalizes mismatches larger than ϵ between the actual and desired array responses for the given angle grid. Therefore, the procedure can be interpreted as a regression problem where the parameter ϵ acts as a sidelobe control parameter while, at the same time, allows some tolerance in the array response for the assumed signal arrival angle. The optimal values of C and ϵ must be obtained for each scenario, depending on the number of sensors, the noise level, the required sidelobe level and the presumed DOA estimation error.

In terms of real variables we can write

$$\mathbf{w}^H\mathbf{R}_x\mathbf{w} = \widetilde{\mathbf{w}}^T\widetilde{\mathbf{R}}_x\widetilde{\mathbf{w}}$$

where

$$\widetilde{\mathbf{w}}^T = \begin{bmatrix}\mathbf{w}_R^T & \mathbf{w}_I^T\end{bmatrix}$$

and

$$\widetilde{\mathbf{R}}_x = \begin{bmatrix} Re(\mathbf{R}_x) & -Im(\mathbf{R}_x) \\ Im(\mathbf{R}_x) & Re(\mathbf{R}_x) \end{bmatrix}.$$

Likewise, the beamformer output for each DOA can be written in terms of real variables as

$$\mathbf{w}^H\mathbf{a}(\theta_i) = \widetilde{\mathbf{w}}^T\widetilde{\mathbf{a}}(\theta_i) + j\widetilde{\mathbf{w}}^T\widetilde{\mathbf{a}}'(\theta_i) \tag{7}$$

where $\widetilde{\mathbf{a}}(\theta_i)$ and $\widetilde{\mathbf{a}}'(\theta_i) \in R^{2M\times 1}$ are given by

$$\begin{aligned} \widetilde{\mathbf{a}}(\theta_i)^T &= \begin{bmatrix}\mathbf{a}_R^T(\theta_i) & \mathbf{a}_I^T(\theta_i)\end{bmatrix} \\ \widetilde{\mathbf{a}}'(\theta_i)^T &= \begin{bmatrix}\mathbf{a}_I^T(\theta_i) & -\mathbf{a}_R^T(\theta_i)\end{bmatrix}. \end{aligned} \tag{8}$$

For notational simplicity, we define the following compact variable $\bar{\mathbf{a}} \in R^{2M\times 2P}$,

$$\bar{\mathbf{a}}(i) = \begin{cases} \widetilde{\mathbf{a}}(\theta_i), & i = 1, \cdots P, \\ \widetilde{\mathbf{a}}'(\theta_{i-P}), & i = P+1, \cdots 2P. \end{cases}$$

Hence, the initial complex formulation (4) can be written in terms of real variables as

$$J(\widetilde{\mathbf{w}}) = \frac{1}{2}\widetilde{\mathbf{w}}^T\widetilde{\mathbf{R}}_x\widetilde{\mathbf{w}} + C\sum_{i=1}^{2P}|y_i - \widetilde{\mathbf{w}}^T\bar{\mathbf{a}}(i)|_\epsilon \tag{9}$$

where the real variable $y_i = Re(d_i)$ for $i = 1, \cdots, P$, and $y_i = Im(d_i)$ for $i = P+1, \cdots, 2P$, represents the desired output for each product $\widetilde{\mathbf{w}}^T\bar{\mathbf{a}}(i)$.

Introducing a set of slack positive variables ξ and $\tilde{\xi}$, the cost function (9) can be written as the following optimization problem with constraints [6]: minimize

$$L(\widetilde{\mathbf{w}}, \boldsymbol{\xi}, \tilde{\boldsymbol{\xi}}) = \frac{1}{2}\widetilde{\mathbf{w}}^T \widetilde{\mathbf{R}}_x \widetilde{\mathbf{w}} + C\sum_{i=1}^{2P}(\xi_i + \tilde{\xi}_i) \tag{10}$$

subject to

$$\begin{aligned} \widetilde{\mathbf{w}}^T \bar{\mathbf{a}}(i) - y_i &\leq \epsilon + \xi_i, & (11)\\ y_i - \widetilde{\mathbf{w}}^T \bar{\mathbf{a}}(i) &\leq \epsilon + \tilde{\xi}_i, & (12)\\ \xi_i, \tilde{\xi}_i &\geq 0 & (13) \end{aligned}$$

for $i = 1, \cdots, 2P$.

4. IRWLS-PROCEDURE

Introducing the constraints into (10), the solution of the optimization problem is a saddle point of the following Lagrange function [6]

$$\begin{aligned} L(\widetilde{\mathbf{w}}, \boldsymbol{\xi}, \tilde{\boldsymbol{\xi}}, \boldsymbol{\alpha}, \tilde{\boldsymbol{\alpha}}, \boldsymbol{\gamma}, \tilde{\boldsymbol{\gamma}}) = \frac{1}{2}\widetilde{\mathbf{w}}^T \widetilde{\mathbf{R}}_x \widetilde{\mathbf{w}} - \sum_{i=1}^{2P}\left(\tilde{\gamma}_i \tilde{\xi}_i + \gamma_i \xi_i\right) \\ + C\sum_{i=1}^{2P}\left(\xi_i + \tilde{\xi}_i\right) - \sum_{i=1}^{2P}\alpha_i\left(y_i - \widetilde{\mathbf{w}}^T \bar{\mathbf{a}}(i) + \epsilon + \xi_i\right) \\ - \sum_{i=1}^{2P}\tilde{\alpha}_i\left(\widetilde{\mathbf{w}}^T \bar{\mathbf{a}}(i) - y_i + \epsilon + \tilde{\xi}_i\right), \end{aligned} \tag{14}$$

minimum with respect to the primal variables $\widetilde{\mathbf{w}}$, ξ_i and $\tilde{\xi}_i$; and maximum with respect to the Lagrange multipliers $\alpha_i \geq 0$, $\tilde{\alpha}_i \geq 0$, $\gamma_i \geq 0$ and $\tilde{\gamma}_i \geq 0$, for $i = 1, \cdots, 2P$.

Similarly to other SVM-based problems, here the optimal beamformer can be expanded in terms of a set of steering vectors, which are the support vectors for the problem. However, due to the output power term in the cost function, now the support vectors in the expansion are transformed by the inverse of the autocorrelation matrix; in particular we have

$$\widetilde{\mathbf{w}} = \sum_{i=1}^{2P}(\alpha_i - \alpha_i)\widetilde{\mathbf{R}}_x^{-1}\bar{\mathbf{a}}(i).$$

The support vectors and their corresponding Lagrange multipliers are typically found by means of quadratic programming (QP) techniques. Recently, it has been shown [7] that the high computational cost of the original procedure can be reduced by transforming the QP problem into an equivalent least squares problem [7]. By applying an iterative re-weighted least square (IRWLS) algorithm the requirements both in time and memory are reduced without any loss of performance [8].

To obtain an IRWLS procedure, we apply the Karush-Kuhn-Tucker (KKT) conditions, which impose that the terms depending on ξ_i and $\tilde{\xi}_i$ must be removed at the solution [7]. Therefore, the cost function (14) can be written as

$$\begin{aligned} L(\widetilde{\mathbf{w}}, \boldsymbol{\alpha}, \tilde{\boldsymbol{\alpha}}) &= \frac{1}{2}\widetilde{\mathbf{w}}^T \widetilde{\mathbf{R}}_x \widetilde{\mathbf{w}} - \sum_{i=1}^{2P}\alpha_i\left(y_i - \widetilde{\mathbf{w}}^T \bar{\mathbf{a}}(i) + \epsilon\right) \\ &\quad - \sum_{i=1}^{2P}\tilde{\alpha}_i\left(\widetilde{\mathbf{w}}^T \bar{\mathbf{a}}(i) - y_i + \epsilon\right) \\ &= \frac{1}{2}\widetilde{\mathbf{w}}^T \widetilde{\mathbf{R}}_x \widetilde{\mathbf{w}} - \frac{1}{2}\sum_{i=1}^{2P}\left(f_i e_i^2 + \tilde{f}_i \tilde{e}_i^2\right), \end{aligned} \tag{15}$$

where

$$\begin{aligned} e_i &= \epsilon + y_i - \widetilde{\mathbf{w}}^T \bar{\mathbf{a}}(i), & f_i &= \frac{2\alpha_i}{\epsilon + y_i - \widetilde{\mathbf{w}}^T \bar{\mathbf{a}}(i)}, \\ \tilde{e}_i &= \epsilon - y_i + \widetilde{\mathbf{w}}^T \bar{\mathbf{a}}(i), & \tilde{f}_i &= \frac{2\tilde{\alpha}_i}{\epsilon - y_i + \widetilde{\mathbf{w}}^T \bar{\mathbf{a}}(i)}. \end{aligned}$$

Observe that (15) can be understood as a weighted least square equation with e_i and $\tilde{e}_i$ as prediction errors and f_i and $\tilde{f}_i$ are its corresponding weights. The minimization of (15) must be carried out iteratively [7].

According to the Representer Theorem [6][7], it can be shown that the coefficients $\widetilde{\mathbf{w}}$, which minimize (14), can be expressed as

$$\widetilde{\mathbf{w}} = \sum_{i=1}^{2P}\left(\tilde{\beta}_i - \beta_i\right)\widetilde{\mathbf{R}}_x^{-1}\bar{\mathbf{a}}(i) = \widetilde{\mathbf{R}}_x^{-1}\Phi^T\gamma, \tag{16}$$

where Φ and γ are given by

$$\begin{aligned} \Phi &= [\bar{\mathbf{a}}(1), \bar{\mathbf{a}}(2), \ldots, \bar{\mathbf{a}}(2P)]^T & (17)\\ \gamma &= \left[\beta_1 - \tilde{\beta}_1, \beta_2 - \tilde{\beta}_2, \ldots, \beta_{2P} - \tilde{\beta}_{2P}\right]^T. & (18) \end{aligned}$$

When the IRWLS procedure converges, the variables $\tilde{\beta}_i$ and β_i achieve the same values as the Lagrange multipliers $\tilde{\alpha}_i$ and α_i.

Substituting (16) into (15) the minimum of the cost function with respect to γ for fixed f_i and $\tilde{f}_i$ is found by solving the linear equation system [7]

$$\left[\mathbf{H} - \mathbf{D}^{-1}_{\tilde{\mathbf{f}}+\mathbf{f}}\right]\gamma = [\mathbf{y} - \boldsymbol{\epsilon}\mathbf{E}] \tag{19}$$

where

$$\begin{aligned} (\mathbf{H})_{i,j} &= \bar{\mathbf{a}}(i)^T \widetilde{\mathbf{R}}_x^{-1}\bar{\mathbf{a}}(j), \quad i,j = 1,\ldots,n \\ \left(\mathbf{D}_{\tilde{\mathbf{f}}+\mathbf{f}}\right)_{i,j} &= \delta[i-j](\tilde{f}_i + f_i), \quad i,j = 1,\ldots,n \\ \mathbf{y} &= [y_1, y_1, \ldots, y_{2P}]^T, \\ \mathbf{E} &= \left[\frac{\tilde{f}_1 - f_1}{\tilde{f}_1 + f_1}, \frac{\tilde{f}_2 - f_2}{\tilde{f}_2 + f_2}, \ldots, \frac{\tilde{f}_{2P} - f_{2P}}{\tilde{f}_{2P} + f_{2P}}\right]^T. \end{aligned}$$

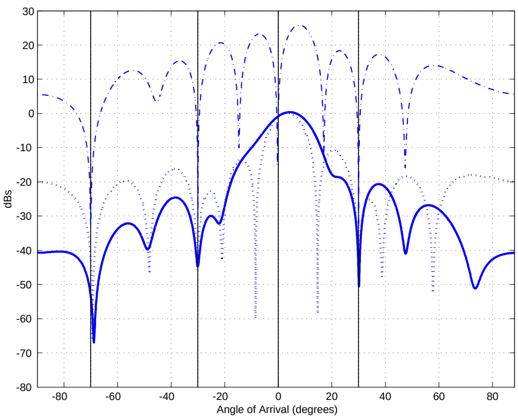

Fig. 1. Direction mismatch of 2^o: SVM (solid line), SpheRCB (dotted line) and Capon (dash-dotted line).

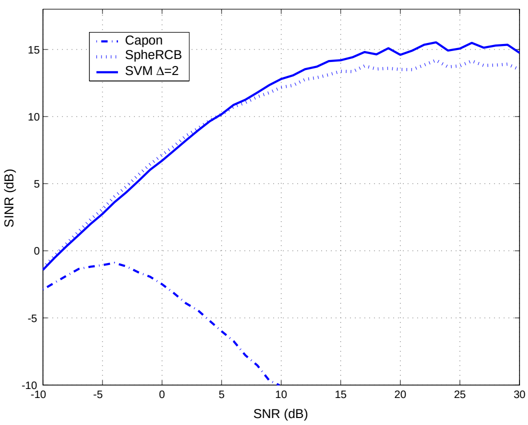

Fig. 2. Direction mismatch of 2^o: SINR versus SNR.

Finally, since f_i and $\tilde{f}_i$ depend on the Lagrange multipliers α_i and $\tilde{\alpha}_i$, we can substitute them using the KKT conditions

$$f_i = \begin{cases} 0 & \text{if } e_i < 0, \\ \frac{2C}{e_i} & \text{if } e_i \geq 0, \end{cases} \tag{20}$$

and equally for $\tilde{f}_i$.

5. SIMULATION RESULTS

To evaluate the performance of the proposed beamforming technique, some computer experiments are carried out in scenarios with source steering vector mismatches. We assume a uniform linear array with $M = 10$ sensors and half-wavelength sensor spacing. All signal waveforms are i.i.d. QPSK. Spatially white Gaussian noise is assumed ($\mathbf{Q} = \sigma_n^2\mathbf{I}$). The power of the signal of interest (SOI) is $\sigma_s^2 = 10$ dB, and the power of the interferences is $\sigma_j^2 = 30$ dB, $\forall j$. The actual source DOA is $\theta_s = 0^o$ and the DOAs of the interferences are $\theta_1 = -30^o$, $\theta_2 = 30^o$ and $\theta_3 = 70^o$. The signal-to-noise ratio SNR is 10 dB. In order to compute $\hat{\mathbf{R}}_x$, $N = 50$ snapshots are used. In all scenarios, each point is the average from 200 simulations.

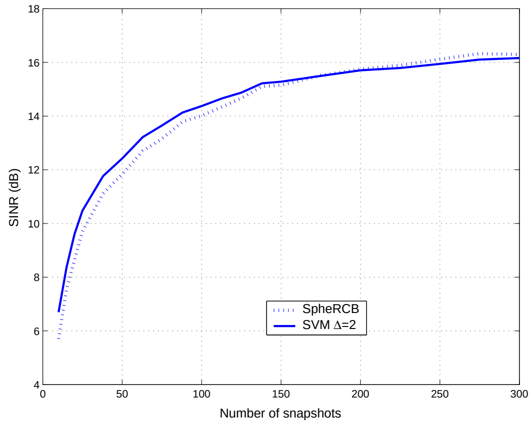

Fig. 3. Direction mismatch of 2^o: SINR versus N.

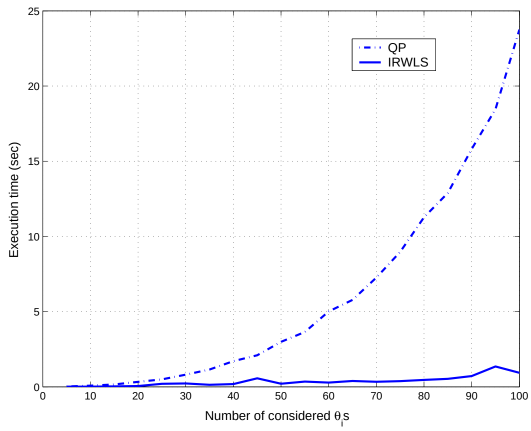

Fig. 4. Computational cost of QP and IRWLS.

Example I

In the first scenario, the presumed source signal DOA is $\theta_a = 2^o$. Defining an angular beamwidth around θ_a of $\triangle = 2^o$, our IRWLS approach employs an uniform grid with $P_1 = 20$ DOAs inside the mainlobe and $P_2 = 40$ outside the mainlobe. Fig. 1 shows a single realization of the beampatterns of the proposed SVM approach ($C = 1$ and $\epsilon = 0.001$) compared to the Capon beamformer and the robust array beamformer with spherical constraint proposed in [1] with control parameter $\epsilon = 4.5$ (denoted as SpheRCB). It is demonstrated in [1] that the SpheRCB beamformer is equivalent to the SOC method proposed in [2]. Clearly, we observe that the SOI is considered to be an interference by the Capon beamformer. On the other hand the source signal is preserved by the SVM and the SpheRCB approaches, but the former has lower sidelobe level.

Fig. 2 depicts the output SINR versus the input SNR. As can be seen, the SVM approach behaves similar to the SpheRCB beamformer in the low SNR region, but the former achieves better performance for high SNR. Fig. 3 shows the SINR versus the number of snapshots N, used to esti-

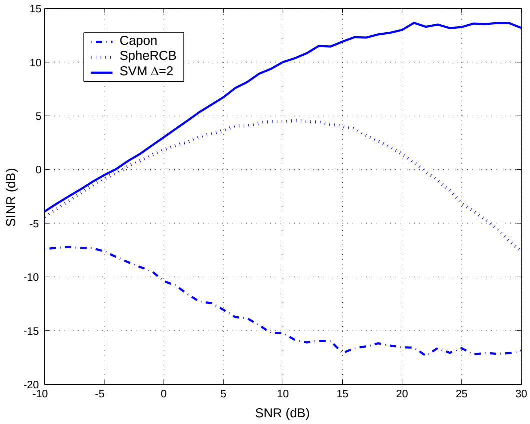

Fig. 5. Direction mismatch of 5^o: SINR versus SNR.

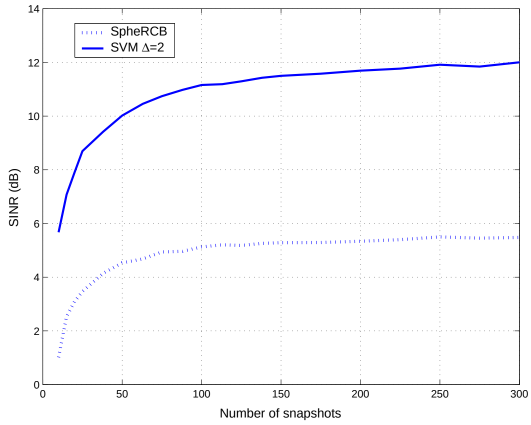

Fig. 6. Direction mismatch of 5^o: SINR versus N.

mate $\hat{\mathbf{R}}_x$: our beamformer operates slightly better than the SpheRCB when the number of snapshots N is less than 150. On the contrary, if N increases, the SpheRCB algorithm outperforms the SVM-based beamformer. Hence, we can conclude that the proposed beamformer is appropriate for scenarios when the number of available snapshots is scarce.

Compared to QP techniques [5], as shown in Fig. 4, an IRWLS procedure reduces considerably the computational cost of the SVM problem. This reduction is especially noticeable when the number of considered angles P increases.

Example II

In the second example, we simulate a scenario where the error in the source DOA is underestimated for both the SVM and SpheRCB beamformer. The presumed source DOA is $\theta_s = 5^o$ (the actual one is $\theta_s = 0^o$). For the SpheRCB approach, we use $\epsilon = 4.5$, which is not large enough for an error of 5^o in the source signal DOA [1]. For the SVM beamformer, the maximum error is controlled by the mainlobe beamwidth, which is chosen as $\triangle = 2^o$.

Figs. 5 and 6 are similar to Figs. 2 and 3. In both examples, the performance of the SpheRCB beamformer degrades considerably; however, the SVM beamformer still achieves good results even when the maximum mismatch error is underestimated. This increased robustness can be also explained by looking at the broad mainlobe provided by the SVM beamformer in Fig. 1.

6. CONCLUSIONS

The SVM-beamformer has been recently proposed as a way to achieve increased robustness against the signal mismatch problem as well as sidelobe control. In this paper the high computational cost of the original SVM solution has been significantly reduced by transforming the original QP problem into an equivalent reweighted least squares problem. The robust performance of the proposed beamformer, even when the mismatch error is underestimated, has been demonstrated by computer simulations.

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7. REFERENCES

[1] J. Li, P. Stoica, and Z. Wang, “On robust capon beamforming and diagonal loading,” *IEEE Trans. Sig. Proc.*, vol. 51, no. 7, pp. 1702–1715, July 2003.

[2] S. A. Vorobyov, A. B. Gershman, and Z. Lou, “Robust adaptive beamfoming using worst-case performance optimization: A solution to the signal mismatch problem,” *IEEE Trans. Sig. Proc.*, vol. 51, no. 2, pp. 313–324, Feb 2003.

[3] D. Feldman and L. J. Griffiths, “A projection approach for robust adaptive beamforming,” *IEEE Trans. Sig. Proc.*, vol. 42, no. 4, pp. 867–876, April 1994.

[4] L. Liu, A. B. Gershman, S. Lou, and S. M. Wong, “Adaptive beamforming with sidelobe control: A second-order cone programming approach,” *IEEE Sig. Proc. Letters*, vol. 10, no. 11, pp. 331–334, Nov 2003.

[5] C. C. Gaudes, J. Vía, and I. Santamaría, “Robust array beamforming with sidolobe control using support vector machines,” in *Proc. SPAWC 2004, Lisboa, Portugal*, July 2004.

[6] V. Vapnik, *The Nature of Statistical Learning Theory*, Springer Verlag, New York, 1995.

[7] F. Pérez-Cruz, A. Navia-Vázqued, P. Alarcón-Diana, and A. Artés-Rodríguez, “An IRWLS procedure for SVR,” in *Proc. EUSIPCO 2000, Tampere, Finland*, Sept. 2000.

[8] F. Pérez-Cruz, C. Bousoño-Calzón, C. H. Lin, and A. Artés-Rodríguez, “Convergence of the irwls procedure to the support vector machine solution,” *submitted to Neural Computation*, 2003.